STAT 582 Homework 5

Due date: In class on Friday, March 18, 2005

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14. (a) Let X_n be independent, Gaussian r.v. with

$$\mathbb{E}[X_n] = 0 \qquad \text{var}(X_n) = \sigma_n^2.$$

Show that $\sum_n X_n$ converges a.s. iff $\sum_n \sigma_n^2 < \infty$.

Hint: Use the Three Series Theorem only in one direction. Using other results might simplify the proof of the opposite direction. You should not need to do any computation: the simple answer is one line in each direction.

(b) Let Y_n be independent, Gaussian r.v. with

$$\mathbb{E}[Y_n] = \mu_n \qquad \text{var}(Y_n) = \sigma_n^2.$$

Show that $\sum_n Y_n$ converges a.s. iff $\sum_n \sigma_n^2 < \infty$ and $\sum_n \mu_n$ converges.

Hint: As above, use results other than the Three Series Theorem to proceed in one direction. Use also (a). Again, you should not need to estimate any integrals, though you can of course proceed in that way.

15. Let B be a symmetrical Binomial r.v. with $P[B = 1] = P[B = -1] = 1/2$. Consider $X_n = (-1)^n B$. Consider the following statements.

(a) The sequence X_n does not converge in probability.

(b) Let $X_{n(k)}$ denote any subsequence. Then, the subsequence $\{n(k)\}_k$ must contain either infinitely many even indices or infinitely many odd indices. Thus, we may choose a subsubsequence $n(k(i))$ for which $X_{n(k(i))}$ converges almost surely.

(c) If every subsequence $X_{n(k)}$ contains a subsubsequence $X_{n(k(i))}$ which converges almost surely, then the sequence X_n converges in probability.

Clearly, not all statements can be true. Indeed, assume that (b) and (c) where both true; that would imply that X_n does converge in probability and so (a) would have to be false. So, if (a) was true, then at least one of (b) and (c) has to be false. State for each statement whether it is true or false; prove your claim for each statement.

16. Let $f_n(x) = 1 - \cos(2n\pi x)$ for $0 < x < 1$ and zero otherwise.

(a) Show that for Lebesgue-almost all $0 < x < 1$ we have that $f_n(x)$ does not converge.[1]

(b) Show that $F_n(t) = \int_{-\infty}^{t} f_n(x)dx$ converges vaguely to the uniform distribution on the unit interval, i.e., to the df F with $F(t) = t$ for $0 \le t \le 1$. Hint: show that if the sequence of df G_n converges to the df G on a dense set then G_n converges to G weakly. To this end, let x be a point where G is continuous and let $\varepsilon > 0$. Show, that there exists m^* such that $G(x) - \varepsilon \le G_m(x) \le G(x) + \varepsilon$ for all $m \ge m^*$.

(c) Does F_n converge to F in total variation? Show your argument.

[1] Note that one should avoid saying that "f_n does not converge Lebesgue-almost everywhere", which means that it is false that f_n converges Lebesgue-almost everywhere. One should avoid it, since it may easily be confused with "f_n does not converge, Lebesgue-almost everywhere", where the coma indicates that we mean here that "Lebesgue-almost everywhere f_n does not converge".