STAT 582 Homework 3

Due date: In class on Monday, February 19, 2007

or slide under door of Instructor's office, Duncan Hall 2082
Instructor: Dr. Rudolf Riedi

9. [A Kinchine-type L_1-LLN. This could be an exam question.]

   Let $\{X_n\}_n$ be a sequence of iid random variables. Set $\bar{X}_n = (1/n) \sum_{k=1\dots n} X_k$.

   We assume that there exists $p \geq 1$ which will be specified such that $X_1 \in L_p$.

   We note that

$$||\bar{X}_n||_p = \left\| \frac{X_1 + \dots X_n}{n} \right\|_p \leq \frac{1}{n} \left(\|X_1\|_p + \dots \|X_n\|_p \right) \tag{1}$$

   (a) Assume $p > 1$. Show that $\{\bar{X}_n\}_n$ is u.i.
   Hint: Use the "triangular inequality" (1).

   (b) Assume $p > 1$. Show that $\bar{X}_n \stackrel{L_1}{\to} \mathbb{E}[X_1]$.
   Hint: You can use the following fact without proof: If X_n are iid and L_1 then $\bar{X}_n \stackrel{\text{a.s.}}{\to} \mathbb{E}[X_1]$.

   (c) Assume $p > 1$. Strengthen your argument above to show that $\bar{X}_n \stackrel{L_q}{\to} \mathbb{E}[X_1]$ for any $q < p$.

   (d) Voluntary: (you do not need to submit this problem for full credit.) Assume $p = 1$. Show that $\bar{X}_n$ converges in L_1.
   Hints: (i) Show that $\bar{X}_n$ has uniformly bounded first moments using the triangular inequality.
   (ii) Show that $\bar{X}_n$ is uniformly absolutely continuous. Use the triangular inequality again and also that the "family" $\{X_1\}$ is u.i., thus, uniformly absolutely continuous.
   (iii) Use again the fact that $\bar{X}_n \stackrel{\text{a.s.}}{\to} \mathbb{E}[X_1]$.

10. [Using moment conditions for u.i. and convergence in L_p.]

    Let $\{Z_n\}_n$ be a sequence of exponential random variables with mean one, i.e., $P[Z_n > x] = \exp(-x)$. Let $\{\lambda_n\}_n$ be a sequence of strictly positive numbers. Set $X_n = Z_n/\lambda_n$; then, obviously, X_n is exponential with $P[X_n > x] = \exp(-x\lambda_n)$.

    (a) Assume that X_n is u.i. Conclude that there exists a constant $\theta > 0$ such that $\lambda_n \geq \theta$ for all n; we say that "the sequence λ_n is bounded away from zero".

    (b) Vice versa, assume that the sequence λ_n is bounded away from zero. Show that X_n is then u.i.
    Hint: Verify a moment condition using that the Z_n have identical distribution.

    (c) Assume that X_n converges in L_p for some $p \geq 1$. Show that the sequence λ_n converges to a positive, non-zero number or diverges to ∞.
    Hint: Consider the mean.

    (d) Assume that Y_n are non-negative random variables. Show that $T_n = Y_1 + \dots + Y_n$ converges in L_1 iff $\sum_n \mathbb{E}[Y_n] < \infty$. Conclude that for exponential variables X_n as above we have that $\sum_n X_n$ converges in L_1 iff $\sum_n 1/\lambda_n < \infty$.
    Hint: Cauchy criterium as for the earlier similar problem about L_2.

    Note: In this example you can verify explicitly that convergence in L_1 implies u.i. (Indeed, if the strictly positive sequence λ_n goes to a positive, non-zero number or diverges to ∞ then certainly it is bounded away from zero.)