

STAT 582 Homework 7

Due date: In class on Wednesday, April 4, 2007

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22. Let $\{X_n\}_n$ be a sequence of Gaussian random variables with means μ_n and variances σ_n^2 .
- (a) Assume that $X_n \xrightarrow{D} X$. Show that necessarily $\mu_n \rightarrow \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \rightarrow \sigma^2$ for some $\sigma^2 \geq 0$.
Hint: First consider $\rho_n := |\phi_n(1)|$; argue that it must converge. Next show it cannot converge to zero. Conclude that σ_n^2 converges to a real non-negative number.
Second, turn to μ_n and note that it is the median of the Gaussian r.v. X_n , i.e., $\mu_n = F_n^{\leftarrow}(1/2)$.
 - (b) Assume that $\mu_n \rightarrow \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \rightarrow \sigma^2$ for some $\sigma^2 \geq 0$. Show that X_n converges in distribution and describe the limiting distribution in terms of μ and σ^2 . Pay attention to the case $\sigma = 0$.
 - (c) Give an example of a sequence X_n as above which does not converge in probability.
Hint: Think what independence would do to the sequence X_n . Also, your example can not have $\sigma^2 = 0$; why?
 - (d) Assume now that there is a sequence of independent Gaussian r.v. Y_k and consider $X_n = Y_1 + \dots + Y_n$ with the notation from above. Clearly, here σ_n^2 is non-decreasing. Assume that $\mu_n \rightarrow \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \rightarrow \sigma^2$ for some $\sigma^2 \geq 0$. Show that X_n converges almost surely.

Hint: Continuity and Uniqueness theorem (see Resnick or class notes).

Note: we showed that a *sum of independent* random variables with *Gaussian* distribution converges almost surely iff it converges in distribution iff the sums of means and variances both converge.

Note: we also showed that a *sequence of independent* random variables with *Gaussian* distribution converges in probability iff it converges in distribution to a *constant* iff the means converge and the variances decay to zero. [In the voluntary problem below we study the question whether the sequence could converge even almost surely.]

23. Let $\{B_k\}_n$ be a sequence of independent Bernoulli random variables with $P[B_k = 1] = 1 - P[B_k = 0] = p_k$. Let $M_n = B_1 + \dots + B_n$ be the associated Binomial variable.
- (a) Show that the characteristic function of M_n is $\phi_n(t) = \prod_{k=1}^n (1 + p_k(e^{it} - 1))$.
 - (b) Fix $\lambda > 0$. For every given n define M_n as above with $p_k = \lambda/n$ ($k = 1, \dots, n$). Now let $n \rightarrow \infty$. Show that $M_n \xrightarrow{D} Y$ for some Poisson r.v. Y . What is the mean of Y ?
 - (c) Assume that $p_n = 1/2$ for all n . Let $X_n = 2B_n - 1$.
 - i. Show that the characteristic function of X_n is $\cos(t)$.
 - ii. Let $S_n = X_1 + \dots + X_n = 2M_n - n$. Show that the characteristic function of S_n/\sqrt{n} is $\phi_n(t) := \cos^n(t/\sqrt{n})$
 - iii. Use the approximation $\cos(t) = 1 - t^2/2$ to show that ϕ_n converges to $\exp(-t^2/2)$. Conclude that S_n/\sqrt{n} converges in distribution to a standard normal. Note: this argument is meant to verify the CLT in this special setting. Don't use the CLT for your argument.

- (d) Why are (b) and (c) not in violation of the convergence to types theorem? (After all, (b) and (c) provide different distributional limits of renormalizations of M_n .)

24. [Borel-Cantelli; on popular demand; voluntary]

Let $\{X_n\}_n$ be independent zero mean Gaussian variables with variances $\sigma_n^2 > 0$. We know that $X_n \xrightarrow{D} X$ iff $\sigma_n^2 \rightarrow \sigma^2 \geq 0$.

(a) Assume $X_n \xrightarrow{P} X$. Show that $\sigma^2 \rightarrow 0$ and $X = 0$ a.s.

(b) Assume that $\sigma^2 \rightarrow 0$. Show that $X_n \xrightarrow{P} 0$.

(c) Assume that $\sigma_n = 1/n$. Show that $X_n \xrightarrow{\text{a.s.}} 0$.

Hint: Let Y be standard normal. Note that the distributions of X_n and Y are connected in a simple way. Apply Borel-Cantelli and Markov inequality.

(d) Assume that $\sigma_n \geq \sqrt{2/\log(n)}$. Show that almost surely $X_n(\omega)$ does not converge in \mathbb{R} . Thus, $X_n \not\xrightarrow{\text{a.s.}} 0$, but $X_n \xrightarrow{P} 0$ is possible (i.e., if $\sigma_n \rightarrow 0$).

Hint: Let Y be standard normal. Then the following estimate might prove useful for $a > 1$; note that $(a+1)^2 < (2a)^2 = 4a^2$ and so:

$$P[|Y| > a] = \frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-t^2/2} dt \geq \frac{2}{\sqrt{2\pi}} \int_a^{a+1} e^{-t^2/2} dt \geq \frac{2}{\sqrt{2\pi}} e^{-(a+1)^2/2} \geq \frac{2}{\sqrt{2\pi}} e^{-2a^2}$$