ELEC 533 Homework 3

Due date: In class on Friday, September 28th, 2001

Instructor: Dr. Rudolf Riedi

11. We return to the simplified Roulette (without "zero"): $\Omega = \{1, \ldots, 36\}$, $P[\{n\}] = 1/36$ for all $n \in \Omega$. There are 3 events we are interested in: E are the even numbers, R are the red numbers, and F are the numbers in the first row, i.e. $F = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34\}$. Unlike true roulette let us assume that the red numbers are $R = \{1, 2, 3, 7, 8, 12, 13, 14, 15, 19, 20, 24, 25, 26, 27, 31, 32, 36\}$:

1	4	**7**	10	**13**	16	**19**	22	**25**	28	**31**	34
2	5	**8**	11	**14**	17	**20**	23	**26**	29	**32**	35
3	6	9	**12**	**15**	18	21	**24**	**27**	30	33	**36**

Consider the random variables X, Y and Z given by:

$$X(\omega) = \begin{cases} 4 & \text{if } \omega \text{ is even} \\ 0 & \text{else} \end{cases} \qquad Y(\omega) = \begin{cases} 7 & \text{if } \omega \text{ is red} \\ 0 & \text{else} \end{cases} \qquad Z(\omega) = \begin{cases} -3 & \text{if } \omega \text{ is in the first row} \\ 0 & \text{else} \end{cases}$$

(a) Compute the marginal distributions of X, Y and Z, i.e., compute $P[X = t]$, $P[Y = t]$, $P[Z = t]$ for all $t \in \mathbb{R}$

(b) Compute the pairwise joint marginal distributions of (X, Y), (X, Z) and (Y, Z), i.e., compute $P[X = s \text{ and } Y = t]$, $P[X = s \text{ and } Z = t]$, $P[Y = s \text{ and } Z = t]$, for all $s, t \in \mathbb{R}$.

(c) Compute the full joint marginal distributions of (X, Y, Z), i.e., compute $P[X = s \text{ and } Y = t \text{ and } Z = u]$ for all $s, t, u \in \mathbb{R}$.

(d) Are the random variables X and Y independent?

(e) Are the random variables X and Z independent?

(f) Are the random variables Y and Z independent?

(g) Are the random variables X, Y and Z independent?

(h) Let us assume that instead of the first row, $Z(\omega)$ equals -3 if the number ω is in the *third* row, i.e. in $\{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$. Of all the questions in 11a to 11g, exactly two have now a different answer. Which are they, and what are the new answers?

The main lesson of this problem is: Given three random variables, it is not enough to check pairwise independence to decide whether the three random variables are independent. Also, the joint distribution F_{XYZ} of three variables can not be computed from the pairwise joint distributions F_{XY}, F_{XZ} and F_{YZ} since different joint distribution functions F_{XYZ} can have the same pairwise joint marginals F_{XY}, F_{XZ} and F_{YZ}.

12. Compute expectation and variance of the following random variables:

(a) $X \simeq \mathcal{U}([0, 2\pi])$: Uniform on $[0, 2\pi]$

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & \text{for } x \text{ in } [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

(b) $X \simeq$ Cauchy:

$$f_X(x) = \frac{1}{\pi(1 + x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

(c) $N \simeq \text{Poiss}(\lambda)$: Poisson with parameter $\lambda > 0$, which is given by

$$P[N = n] = P_N(n) = \begin{cases} e^{-\lambda} \frac{\lambda^n}{n!} & \text{for } n = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(d) $X \simeq \mathcal{N}(\mu, \sigma^2)$: Gaussian or normal distribution with parameters $\sigma^2 > 0$ and $\mu \in \mathbb{R}$, which is given through the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

(You don't have to show that this is indeed a probability density.)

(e) $X \simeq \exp(\lambda)$: One sided exponential with parameter $\lambda > 0$, which is given through the density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

13. Assume that X and Y are independent Gaussian random variables with zero mean and variance 1. Compute the distribution of the random variable $Z = \exp(-(X^2 + Y^2)/2)$.
Hint: the transformation from Cartesian to polar coordinates goes as $x = r\sin(\phi)$, $y = r\cos(\phi)$, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\phi$.

14. Given is a r.v. X which is uniformly distributed on $[0, \pi]$. We are interested in the r.v. $Y = g(X)$ where $g(t) = \sin(t)$.

    (a) Compute $\mathbb{E}[\sin(X)] = \int \sin(x) f_X(x)\mathrm{d}x$.

    (b) Compute the pdf (density) f_Y of Y. (You might find it convenient to compute first the CDF F_Y of Y.)

    (c) Compute $\mathbb{E}[Y] = \int y f_Y(y)\mathrm{d}y$. Check whether you got the same answer as in (a).

15. (a) Let X be a continuous r.v. Using the definition of expectation and the rules of integration derive

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b,$$

where a and b are constants. Also show that

$$\mathbb{E}[(aX + b)^2] = a^2\,\mathbb{E}[X^2] + 2ab\,\mathbb{E}[X] + b^2.$$

Conclude that $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$.

    (b) Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = aX + b$. The mean and variance of Y can be computed using 15a. Find the p.d.f of Y.

    (c) Repeat 15b for $X \sim U(0, 2\pi)$.