ELEC 533 Homework 8

Due date: In class on Wednesday, November 21st, 2001

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29. Recall the definition of the k-th cumulant λ_k as $(-i)^k \cdot \psi^{(k)}(0)$, i.e., $(-i)^k$ times the k-th derivative at $u = 0$ of $\psi(u) = \log \mathbb{E}[\exp(iuX)]$ (the logarithm of the characteristic function). Recall also that Skew is defined as $\lambda_3/(\lambda_2)^{3/2}$ and Kurtosis as $\lambda_4/(\lambda_2)^2$.

    (a) Show that the cumulants of order 3 and higher for a Gaussian r.v. are zero. Thus in particular, the Skew and Kurtosis for a Gaussian r.v. are zero.

    (b) Express the cumulants (of all orders) of an exponential r.v. in terms of the parameter r of its p.d.f.

    $$f(x) = \begin{cases} r \cdot \exp(-rx) & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

    Thereby compute the Skew and Kurtosis for an exponential r.v.

    Note that Skew is zero for a symmetrical distribution. If non zero, the Skew is a measure of asymmetry. The Kurtosis measures the "weight" of the tails: the larger, the heavier the tails.

30. Recall the Large Deviation Principle (LDP) which states that for a sequence of independent, identically distributed random variables X_n we can bound the probability of a deviation of sample means from the true mean as follows. In the case where $\mathbb{E}[X] = 0$:

    $$P[(X_1 + \ldots + X_n)/n > a] \leq \exp\left(n \inf_{q>0} \left(\log \mathbb{E}[e^{qX}] - qa\right)\right)$$

    (a) Formulate a bound in the general case, i.e., if $\mathbb{E}[X] = \mu$.

    (b) Compute $\inf_{q>0}(\log \mathbb{E}[e^{qX}] - qa)$ in the Gaussian case, i.e., for $X_n \sim \mathcal{N}(0,1)$.

    (c) For the same Gaussian case deduce an upper bound of $P[(X_1 + \ldots + X_n)/\sqrt{n} > b]$ and compare this bound to the true probability (which you can write implicitly as an integral using that $X_1 + \ldots + X_n$ is Gaussian). Would you say that the bound provided by the Large Deviation Principle is effective?

    In the general case (iid, finite variance but non-Gaussian r.v. X_n) we may still derive a bound of $P[(X_1 + \ldots + X_n)/\sqrt{n} > b]$ from the LDP. This, in fact, estimates how fat the tails of $(X_1 + \ldots + X_n)/\sqrt{n}$ are. We also know that $(X_1 + \ldots + X_n)/\sqrt{n}$ converges to a Normal law due to the CLT. Comparing the LDP bound with the above bound on the tails for a Gaussian, this provides us, then, with a means to assess the speed of convergence in the CLT.

31. Recall the stable distributions: X is distributed as a symmetrical stable variables, more precisely $X \simeq S\alpha S(\mu, \sigma)$ $(0 < \alpha \leq 2)$, if and only if

$$\Phi_X(u) = e^{(i\mu u - |\sigma|^\alpha |u|^\alpha)},$$

where μ is the position parameter, and σ is the scale parameter.

(a) Assume that $X \sim S\alpha S(0, 1)$. Show that $Y = \sigma X + \mu$ is distributed as $S\alpha S(\mu, \sigma)$. This explains why μ is called the position parameter, and σ is the scale parameter.

(b) Using the rules for the characteristic function show that the sum of two *independent* $S\alpha S$ r.v.'s is also $S\alpha S$. (Note that a Cauchy r.v. is $S\alpha S$ with $\alpha = 1$: so, we solved this problem for the special case $\alpha = 1$ already earlier.) This explains the name "stable". The distributions of these random variables are stable under addition.

(c) Let $X \simeq S\alpha S(\mu, \sigma)$ and assume that $\alpha > 1$. Show that $\mu = \mathbb{E}[X]$. (Note: When $\alpha \leq 1$, then $\mathbb{E}[|X|] = \infty$. Hence $\mathbb{E}[X]$ is **not defined** when $\alpha \leq 1$.) This gives the intuitive interpretation of the position parameter in the case when the stable distribution have a well defined mean.

(d) For the remainder of the homework assume that $\mu = 0$. and let X_n be a sequence of independent, identically distributed symmetrical stable variables, more precisely $X_n \simeq S\alpha S(0, \sigma)$. Use the characteristic function to show that

$$Z_n := \frac{X_1 + \ldots + X_n}{n^{1/\alpha}}$$

are distributed as X_n, i.e. $Z_n \simeq S\alpha S(0, \sigma)$. Conclude that Z_n converges in distribution. For which choices of α does the CLT hold, and for which not?