ELEC 533 Homework 3

Due date: In class on Friday, September 20th, 2002

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10. Given is a r.v. X which is uniformly distributed on $[0, \pi]$. We are interested in the r.v. $Y = g(X)$ where $g(t) = \sin(t)$.
    (a) Compute $\mathbb{E}[\sin(X)] = \int \sin(x) f_X(x) \mathrm{d}x$.
    (b) Compute the pdf (density) f_Y of Y. (You might find it convenient to compute first the CDF F_Y of Y.)
    (c) Compute $\mathbb{E}[Y] = \int y f_Y(y) \mathrm{d}y$. Check whether you got the same answer as in (a).

11. (a) Let X be a continuous r.v. Using the definition of expectation and the rules of integration derive

    $$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b,$$

    where a and b are constants. Also show that

    $$\mathbb{E}[(aX + b)^2] = a^2\,\mathbb{E}[X^2] + 2ab\,\mathbb{E}[X] + b^2.$$

    Conclude that $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$.
    (b) Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = aX + b$. The mean and variance of Y can be computed using 11a. Find the p.d.f of Y.
    (c) Repeat 11b for $X \sim U(0, 2\pi)$.

12. Suppose that X (signal) is a binary r.v. with $P[X = 1] = \alpha$ and $P[X = -1] = \beta = 1 - \alpha$. Suppose the r.v. N is normal, i.e. $N \sim \mathcal{N}(0, \sigma^2)$ (noise). Assume also that N and X are independent, meaning that the events $\{X = a\}$ and $\{N \leq b\}$ are independent for all a and b. We are interested in the r.v. $Y = X + N$ (noisy observation).
    (a) Express $P[Y \leq y | X = 1]$ and $P[Y \leq y | X = -1]$ in terms of Gaussian integrals. (Note that $\int_{-\infty}^{t} \exp(-x^2) dx$ has no closed form.)
    (b) Using this and the law of total probability find $F_Y(y)$, again in terms of integrals.
    (c) Derive a closed form expression for f_Y from F_Y. This is a mixture density; identify it's components in terms of known density functions.
    (d) To infer the signal X from the observation Y the following strategy is used:
        - If $Y \geq \gamma$, we infer that $X = 1$,
        - if $Y < \gamma$, we infer that $X = -1$,

        where γ is some number we will decide on later (next question). We make an error in the inference if $Y \geq \gamma$ and $X = -1$, or if $Y < \gamma$ and $X = 1$. Compute the probability p_e that this happens.
    (e) Find the γ that minimizes the probability of error p_e.

(see reverse side)

13. Compute expectation and variance of the following random variables:

(a) $X \simeq \mathcal{U}([0, 2\pi])$: Uniform on $[0, 2\pi]$

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & \text{for } x \text{ in } [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

(b) $X \simeq$ Cauchy:

$$f_X(x) = \frac{1}{\pi(1 + x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

(c) $N \simeq \text{Poiss}(\lambda)$: Poisson with parameter $\lambda > 0$, which is given by

$$P[N = n] = P_N(n) = \begin{cases} e^{-\lambda} \frac{\lambda^n}{n!} & \text{for } n = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(d) $X \simeq \mathcal{N}(\mu, \sigma^2)$: Gaussian or normal distribution with parameters $\sigma^2 > 0$ and $\mu \in \mathbb{R}$, which is given through the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2} \right).$$

(You don't have to show that this is indeed a probability density.)

(e) $X \simeq \exp(\lambda)$: One sided exponential with parameter $\lambda > 0$, which is given through the density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$