# ELEC 533 Homework 7

Due date: In class on Wednesday, October 29, 2003

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26. Let $X_n$ be Gaussian r.v.-s with mean $\mu_n$ and variance $\sigma_n^2$. Under what conditions on the sequences $\mu_n$ and $\sigma_n^2$ does $X_n$ converge in distribution and what is the limiting distribution. Hint: Use the fact that $X_n$ converges in distribution if and only if their characteristic functions $\phi_{X_n}$ converge. Check, under what conditions the limit of $\phi_{X_n}$ exists and make sure that the limit is again a meaningful characteristic function.

27. Suppose that

$$X_m \xrightarrow{\text{D}} X$$

and that there is a constant $c$ such that $P[X = c] = 1$. Show that

$$X_m \xrightarrow{\text{i.p.}} X$$

[HINT: You need to show that $P[|X_m - X| > \varepsilon] \to 0$ $(m \to \infty)$ for any $\varepsilon$. Because $X$ is essentially constant and equal to $c$, this probability is equal to $1 - P[c-\varepsilon \leq X_m \leq c+\varepsilon]$ which is easily expressed in terms of the CDFs $F_{X_m}$. Now, use that $F_{X_m}$ converges to $F_X$, which has a particularly simple form because $X$ is essentially constant.]

28. Let $X$ be a Bernoulli random variable which takes the values 1 and $-1$ both with probability 1/2.

    (a) Compute the characteristic function of $X$, i.e., $\phi_X(u) = \mathbb{E}[\exp(iuX)]$. Find a simple expression in terms of a cosine function.

    (b) Verify that $\phi''(0) = -\mathbb{E}[X^2]$.

    (c) Compute the characteristic function of

    $$Y_n = \frac{X_1 + \ldots + X_n}{2\sqrt{n}}$$

    using the above simple formula. Here, the random variables $X_n$ are independent and of the same distribution as $X$.

    (d) Approximate the cosine function by its Taylor polynomial of order 2 (i.e., the quadratic polynomial that provides the best approximation) and compute the limit of the characteristic function of $Y_n$. Conclude that $Y_n$ converges in distribution. What is the limiting distribution?

29. Let us define the $k$-th cumulant $\lambda_k$ as

$$\lambda_k := (-i)^k \cdot \psi^{(k)}(0),$$

where $\psi^{(k)}$ denotes the $k$-th derivative of $\psi(u) = \log \mathbb{E}[\exp(iuX)]$ (the logarithm of the characteristic function).

    (a) Show that the cumulants of order 3 and higher for a Gaussian r.v. are zero.

    (b) Express the cumulants (of all orders) of an exponential r.v. in terms of the parameter $r$ of its p.d.f.

    $$f(x) = \begin{cases} r \cdot \exp(-rx) & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

Remark: Beside mean and variance, other quantities are used to describe distributions: Skew is defined as $\lambda_3/(\lambda_2)^{3/2}$ and Kurtosis as $\lambda_4/(\lambda_2)^2$. Note that Skew is zero for a symmetrical distribution. If non zero, the Skew is a measure of asymmetry. The Kurtosis measures the "weight" of the tails: the larger, the heavier the tails. Skew and Kurtosis for a Gaussian r.v. are both zero, thus, a Gaussian has light tails and is symmetrical.