Solutions to Exam 1 (2-21-2017 Stat 419/519):

1. Let B represent the symbol for the unit of barrels of oil. Then

   (a) B for mean

   (b) B^2 for variance

   (c) B^{-1} for the (any) density function. Note:

   — $\int_x f(x)\,dx = 1$ and follow units
   — $U(a,b)$ has density $f(x) = \frac{1}{b-a}$
   — $N(\mu,\sigma^2)$ has density $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-(x-\mu)^2/2\sigma^2)$

2. With the bivariate normal density in standard form, the likelihood is

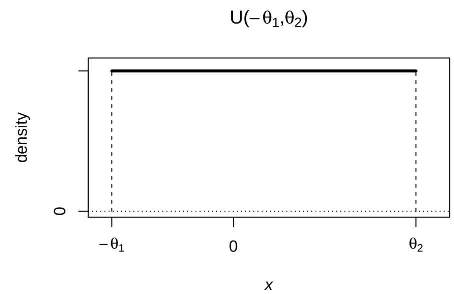
$$\prod_{i=1}^{n} f(x_i, y_i|\rho) = (2\pi)^{-n}(1-\rho^2)^{-n/2} \exp\left[-\frac{1}{2}\sum_i \frac{x_i^2 - 2\rho x_i y_i + y_i^2}{1-\rho^2}\right].$$

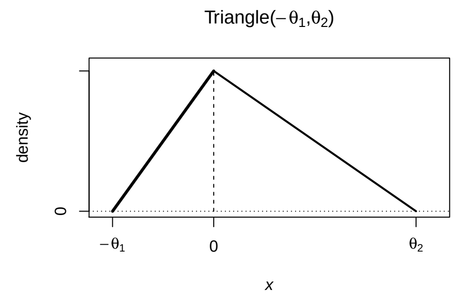
By inspection, the sufficient statistics are

$$\sum x_i^2, \qquad \sum y_i^2, \quad \text{and} \quad \sum x_i y_i .$$

Note that these are 3 separate statistics; adding two doesn't work.

3. Note: We analyzed $U(-\theta, \theta)$ and $Tri(-\theta, \theta)$ in class and examples.





The density functions for the two figures are

$$f(x|\theta_1, \theta_1) = \frac{1}{\theta_1 + \theta_2} I(x \geq -\theta_1) I(x \leq \theta_2)$$

and

$$f(x|\theta_1, \theta_1) = \begin{cases} 0 & x < -\theta_1 \\ \frac{2}{\theta_1(\theta_1+\theta_2)}(x + \theta_1) & -\theta_1 < x < 0 \\ \frac{2}{\theta_2(\theta_1+\theta_2)}(\theta_2 - x) & 0 < x < \theta_2 \\ 0 & x > \theta_2 \end{cases}$$

(a) The sufficient statistics are $X_{(1)}$ and $X_{(n)}$, since

$$\begin{aligned} f(\boldsymbol{x}|\theta_1, \theta_2) &= \prod_{i=1}^{n} f(x_i|\theta_1, \theta_2) \\ &= (\theta_1 + \theta_2)^{-n} \prod_{i=1}^{n} I(x_i \geq -\theta_1) I(x_i \leq \theta_2) \\ &= (\theta_1 + \theta_2)^{-n} I(x_{(1)} \geq -\theta_1) I(x_{(n)} \leq \theta_2) \,. \end{aligned}$$

(b) Follows from the equation

$$\frac{f(\boldsymbol{x}|\theta_1, \theta_2)}{f(\boldsymbol{y}|\theta_1, \theta_2)} = \frac{I(x_{(1)} \geq -\theta_1) I(x_{(n)} \leq \theta_2)}{I(y_{(1)} \geq -\theta_1) I(y_{(n)} \leq \theta_2)} \,.$$

(c) No, since in that case, $f(x_{(1)}) = f(x_{(n)}) = 0$ and the likelihood is 0.

(d) As in many of the homework problems, the full set of order statistics.

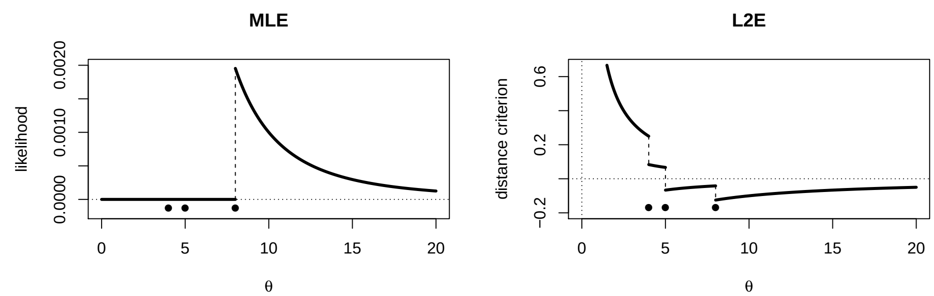
4. Since a sufficient statistic exists, the likelihood has the factorization representation (see Theorem 6.2.6)

$$\begin{aligned}
\hat{\theta}_{\text{MLE}}(\boldsymbol{x}) &= \arg\max_{\theta} f(\boldsymbol{x}|\theta) \\
&= \arg\max_{\theta} g(T(\boldsymbol{x})|\theta)h(\boldsymbol{x}) \\
&= \arg\max_{\theta} g(T(\boldsymbol{x})|\theta)\,.
\end{aligned}$$

Thus the MLE is a function of the data only through the statistic $T(\boldsymbol{x})$.

Note: The sufficient statistics has all the information, but we don't know if the MLE uses it or not? We also don't know if an unbiased estimator exists, so Rao-Blackwell doesn't necessarily apply.

5. Note the at $x = x_{(3)} = 8$, the MLE attains a maximum and the L2E attains a minimum in the graphs.



(a) As we have seen, the MLE of a random sample of size n for the $U(0,\theta)$ density is $X_{(n)}$; hence, the MLE will be the "new" $x_{(3)}$.

(b) The L2E for the $U(0,\theta)$ density is

$$\int_{-\infty}^{\infty} f(x)^2 dx - \frac{2}{n}\sum_{i=1}^{n} f(x_i) = \int_0^{\theta} \frac{1}{\theta^2} dx - \frac{2}{3}\sum_{i=1}^{3}\frac{1}{\theta} I(0 \le x_i \le \theta)$$

$$= \frac{1}{\theta} - \frac{2}{3}\sum_{i=1}^{3}\frac{1}{\theta} I(0 \le x_i \le \theta)\,.$$

For the 3 values $\boldsymbol{x} = (4,5,8)$, this is defined piecewise as

$$\text{L2E}(\theta) = \begin{cases} \frac{1}{\theta} - \frac{2}{3}\cdot\frac{0}{\theta} = \frac{1}{\theta} & 0 \le \theta < 4 \\ \frac{1}{\theta} - \frac{2}{3}\cdot\frac{1}{\theta} = \frac{1}{3\theta} & 4 \le \theta < 5 \\ \frac{1}{\theta} - \frac{2}{3}\cdot\frac{2}{\theta} = -\frac{1}{3\theta} & 5 \le \theta < 8 \\ \frac{1}{\theta} - \frac{2}{3}\cdot\frac{3}{\theta} = -\frac{1}{\theta} & 8 \le \theta \end{cases}$$

as shown in the figure. Note that the minimum value of L2E (which is $-1/x_{(3)} = -1/8$) is attained at $x = x_{(3)} = 8$. The L2E value at $x = x_{(2)} = 5$ is $-1/15$, so if $x_{(3)} > 15$, the L2E minimum occurs at $x = 5$, since $-1/15 < -1/\theta$ when $\theta > 15$.

6. It is easy to verify that

$$f(x) = \frac{1}{b-a} \qquad \mathrm{E}X = \frac{a+b}{2} \qquad \mathrm{Var}X = \frac{(b-a)^2}{12}\,.$$

(519) Letting the first two sample moments be denoted by m and v, we must solve the pair of equations

$$m = \frac{a+b}{2} \qquad \text{and} \qquad v = \frac{1}{12}(b-a)^2\,.$$

The first equation implies that $b = 2m - a$; hence,

$$\begin{aligned} v &= \frac{1}{12}(b-a)^2 \\ &= \frac{1}{12}(2m-a-a)^2 \\ &= \frac{1}{12}(2m-2a)^2 \\ &= \frac{1}{3}(m-a)^2 \\ &= \frac{1}{3}(a-m)^2\,. \qquad \text{Therefore,} \\ a - m &= \pm\sqrt{3v} \quad \text{or} \\ a &= m \pm \sqrt{3v} \quad \text{and} \\ b &= 2m - a \\ &= m \mp \sqrt{3v}\,. \end{aligned}$$

Since $a < b$, we see $(\hat{a}, \hat{b}) = (m - \sqrt{3v}, m + \sqrt{3v})$.

(419) If $a = 0$ is known, then a MoM estimate may be obtained by either solving $m = b/2$ or $\hat{b} = 2m = 2\bar{x}$, or $v = b^2/12$, which implies $\hat{b} = 2\sqrt{3v}$.

7. If $X \sim P(\lambda)$, then $\mathrm{E}X = \mathrm{Var}X = \lambda$. Recall

$$\mathrm{E}\bar{X} = \mathrm{E}X \quad \text{and} \quad \mathrm{Var}\bar{X} = \frac{\mathrm{Var}X}{n}.$$

(a) With $W = a\bar{X}$,

$$\begin{aligned} \mathrm{E}\,W &= a\,\mathrm{E}\bar{X} = a\lambda \\ \mathrm{Var}\,W &= a^2\,\mathrm{Var}\bar{X} = a^2\frac{\lambda}{n}\,; \quad \text{hence,} \\ \mathrm{MSE}(a) &= \mathrm{Var}\,W + (\mathrm{E}\,W - \lambda)^2 \\ &= a^2\frac{\lambda}{n} + (a\lambda - a)^2 \\ &= a^2\frac{\lambda}{n} + (a-1)^2\lambda^2\,. \end{aligned}$$

(b) Optimizing over a not λ,

$$\begin{aligned} \frac{\partial\,\mathrm{MSE}(a)}{\partial a} &= 2a\frac{\lambda}{n} + 2(a-1)\lambda^2\,, \quad \text{which vanishes when} \\ a\left(\frac{\lambda}{n} + \lambda^2\right) &= \lambda^2 \\ a\lambda\left(\frac{1}{n} + \lambda\right) &= \lambda^2 \\ a\left(\frac{1+n\lambda}{n}\right) &= \lambda\,; \quad \text{and finally,} \\ a^* &= \frac{n\lambda}{1+n\lambda}\,. \end{aligned}$$

(c) The optimal risk is given by

$$\begin{aligned} \mathrm{MSE}(a^*) &= \left(\frac{n\lambda}{1+n\lambda}\right)^2\frac{\lambda}{n} + \left(\frac{n\lambda}{1+n\lambda} - 1\right)^2\lambda^2 \\ &= \frac{n\lambda^3}{(1+n\lambda)^2} + \frac{\lambda^2}{(1+n\lambda)^2} \\ &= \frac{(n\lambda+1)\lambda^2}{(1+n\lambda)^2} \\ &= \frac{\lambda^2}{1+n\lambda}\,. \end{aligned}$$

(d) The improvement in risk is given by

$$\frac{\mathrm{MSE}(a^*)}{\mathrm{MSE}(a=1)} = \frac{\lambda^2}{1+n\lambda}\frac{n}{\lambda} = \frac{n\lambda}{1+n\lambda}$$
$$= \frac{1}{1+\frac{1}{n\lambda}} < 1\,,\ \text{approaching 1 as } n \to \infty\,.$$

8. This identity is proven by integrating by parts.