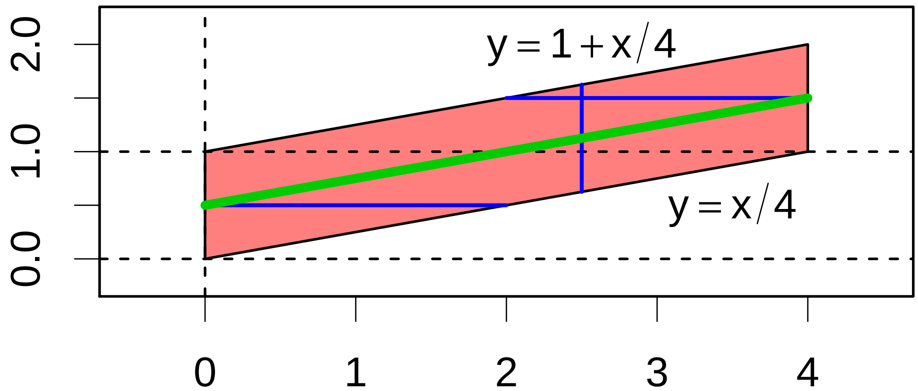
1. Suppose the random variables X and Y follow the bivariate normal PDF. If

$$Var\,(Y \mid X = x) = \frac{1}{4}\,Var\,Y\,,$$

what is the correlation between X and Y? *Hint: You may assume* $\sigma_Y^2 = 1$ *w.l.o.g.*

2. Suppose the joint distribution of the random variables (X, Y) is uniform over the region (parallelogram) shown on the following page.
   (a) Find the marginal PDFs of X and Y. Find μ_x, σ_x^2, μ_y, and σ_y^2.
   (b) Find the covariance and correlation between X and Y.
   (c) Find the conditional distribution of $Y|X = x$. Sketch the conditional mean.
   (d) Find the conditional variance of $Y|X = x$ and compare to the unconditional variance σ_y^2.



3. Suppose we have $n = 3$ random samples from a PDF with mean equal to 0 and finite variance $\sigma^2 > 0$. We compute the sample mean of the first 2 points, as well as all 3 points:

$$\bar{X}_2 = \frac{X_1 + X_2}{2}$$
$$\bar{X}_3 = \frac{X_1 + X_2 + X_3}{3}$$

What is the correlation between $\bar{X}_2$ and $\bar{X}_3$?

4. Suppose the random variable X follows the Rayleigh PDF given by

$$f(x) = x\,e^{-x^2/2} \quad \text{for } x > 0\,.$$

Consider the transformation

$$y = y(x) = x^2\,.$$

Find the PDF of Y, call it $g(y)$, using the change-of-variables technique. Name the PDF.

5. We wish to simulate random numbers from the PDF

$$f(x) = 3x^2\,, \quad \text{for } 0 < x < 1\,.$$

Given a stream of $U(0,1)$ random variables, use the Probability Integral Transform to show how to generate an x given $u \sim U(0,1)$?

6. Consider an infinite sequence of random variables (r.v.'s)

$$X_1, X_2, X_3, \ldots$$

that in fact are *all independent and identically distributed* to the r.v. $X \sim f(x)$. Does the sequence $\{X_i\}$ converge to the random variable X? If so, in what sense? If not, why not?

7. Show that the sample variance S^2 based upon n random samples converges in probability to σ^2. Assume the theoretical variance of S^2,

$$Var\, S^2 = \frac{\kappa - \sigma^4}{n} + \frac{2\sigma^4}{n(n-1)}$$

exists and is finite. (Fill in the details from the book's argument.)

8. Suppose the two random variables, X and Y, are independent and have the PDFs

$$X \sim B(30, p) \quad \text{and} \quad Y \sim B(50, p)\,.$$

Find the PDF of the random variable $X + Y$.

Hints: Use the MGF technique for sums. $B(n,p)$ is the binomial PDF.

9. Suppose the random variable X follows the Poisson $P(m)$ PDF, and that you have a random sample $X_1, X_2, \ldots, X_n$ from it.
   (a) What is the Cramér-Rao Lower Bound on the variance of any unbiased estimator of the parameter m?
   (b) What is the maximum likelihood estimator of m?
   (c) Does the variance of the MLE achieve the CRLB for all n?

10. Consider a random sample of size n from the Beta PDF with parameter $\alpha > 0$ and second parameter $\beta = 1$. Find the sufficient statistic for α.

11. Suppose we have two **unbiased estimators**, X and Y, for an unknown parameter θ. But the variance of Y is greater than the variance of X; hence, X is more efficient. Furthermore, X and Y are *not independent*, but are **correlated**. In fact,

$$\sigma_Y^2 = 4\,\sigma_X^2 \quad \text{and} \quad \rho_{XY} = -\frac{1}{2}\,.$$

Show the following three estimators — $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ — are also unbiased for θ and compute their variance. Order the 3 estimators and determine which is "best?"

(a)

$$\hat{\theta}_1 = X \quad \text{itself?}$$

(b)

$$\hat{\theta}_2 = \frac{X+Y}{2}\,, \quad \text{their average?}$$

(c)

$$\hat{\theta}_3 = \frac{5}{7}X + \frac{2}{7}Y\,, \quad \text{a weighted average?}$$