1. Show that if the random variables X and Y follow the standard Bivariate Normal PDF with single parameter ρ,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-(x^2-2\rho xy+y^2)}{2(1-\rho^2)}},$$

Show that X and Y are independent *if and only if* $\rho = 0$.

Hints: Recall $\mu_X = \mu_Y = 0$ *and* $\sigma_X = \sigma_Y = 1$ *for the standard Bivariate Normal PDF. Note there are 2 parts to this question: to show* $A \Longrightarrow B$ *and then* $A \Longleftarrow B$.

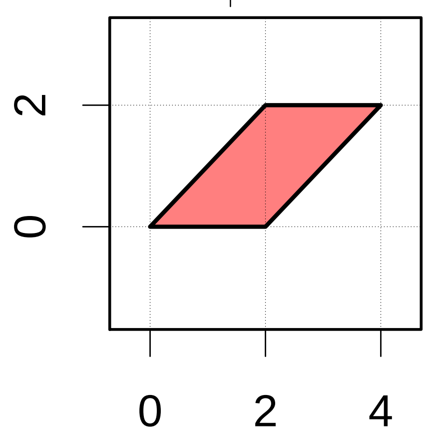
2. Suppose the joint distribution of the random variables (X, Y) is uniform over the region (parallelogram) shown. For parts (b) and (c), include a rough sketch of your answer.

Hints: Each question can be answered without calculus, just "geometric insight." Write the equations for the 4 sides carefully. Sketch the conditional std deviation for part (c).

   (a) For $0 < x < 4$, what is the PDF of the conditional r.v. $Y|X = x$?

   (b) What is the conditional mean $m(x) = E(Y|X = x)$ for $0 < x < 4$?

   (c) What is the conditional variance of $Y|X = x$ for $0 < x < 4$?



3. Suppose we have $n = 3$ random samples from a PDF with mean equal to 0 and finite variance $\sigma_X^2 > 0$. We introduce 2 new random variable derived from X_1, X_2, X_3:

$$Y_1 = X_1 + X_2$$
$$Y_2 = X_1 + X_3$$

What is the **correlation**, ρ, between Y_1 and Y_2?

4. Suppose the random variable X follows a particular Negative Exponential PDF given by

$$f(x) = \frac{1}{2}\, e^{-x/2} \quad \text{for } x > 0\,.$$

Consider the transformation

$$y = y(x) = +\sqrt{x}\,.$$

   (a) Find the PDF of Y, call it $g(y)$, using the change-of-variables technique.

   (b) Can you give the name of this PDF?

5. We wish to simulate random numbers from the PDF

$$f(x) = 4x^3\,, \quad \text{for } 0 < x < 1\,,$$

which is a $Beta(4,1)$ random variable. Given a stream of $Unif(0,1)$ random variables, use the Probability Integral Transform (PIT) to show how to generate an x given $u \sim Unif(0,1)$?

6. Suppose the random variables X and Y are standard bivariate Normal with correlation coefficient ρ. What is the exact PDF of the random variable

$$V = X + Y \ ?$$

Hint: $$E\left[e^{t\,V}\right] = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{t(x+y)} \cdot \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{\frac{-(x^2-2\rho xy+y^2)}{2(1-\rho^2)}} = e^{(1+\rho)t^2}\,.$$

7. Suppose we have a random sample from the $Unif(-\theta,\theta)$, where $\theta > 0$. Note we use the closed interval $[-\theta,\theta]$ for the sample.

   (a) What is the MLE of θ?

   (b) For the random sample of size $n = 10$,

$$(-0.850, 1.153, -0.364, 1.532, 1.762, -1.818, 0.112, 1.570, 0.206, -0.174)\,,$$

   what is $\hat{\theta}_{MLE}$? FYI, generated by `> set.seed(123); round(runif(10,-2,2),3)`.

8. If $\{X_1, X_2, \ldots, X_n\}$ is a random sample from the $N(\mu_X, \sigma_X^2)$ PDF, what is the PDF of the random variable

$$Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu_X}{\sigma_X}\right)^2 \ ?$$

9. The MLE of σ^2 for a $N(\mu,\sigma^2)$ random sample is

$$\tilde{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\,.$$

   (a) Compute the bias of $\tilde{S}^2$.

(b) Show $\tilde{S}^2$ is asymptotically unbiased, that is, the bias vanishes as the sample size $n \to \infty$.

Hint: The bias of $\hat{\theta}(\boldsymbol{X})$ is the difference of $E[\hat{\theta}(\boldsymbol{X})]$ and the true parameter θ.

10. Instead of recording successes and failures a 1's and 0's as with the Bernoulli/Binomial PMF, we use for each trial outcome

$$X_i = \begin{cases} 1 & \text{if success, with probability } p \\ -1 & \text{if failure, with probability } 1-p\,. \end{cases} \tag{1}$$

(a) Let $Y = \sum_{i=1}^{n} X_i$. What does Y measure?

(b) For a Bernoulli ($Binom(1,p)$) trial, we cleverly write

$$p_X(x|p) = p^x(1-p)^{1-x}\,.$$

How can you write the PMF in Equation (1) in a similar compact formula?

(c) What is the sufficient statistic for p?

(d) Find the MLE, $\hat{p}(\boldsymbol{X})$ for p?

(e) What is the variance of $\hat{p}$? *Hint: Find the mean and variance of X_i.*

(f) What is the CRLB? *Hint: $I_1(p)$ uses $\ell(p|X)$, not $\ell(p|\boldsymbol{X})$.*

(g) If there are 20 more heads than tails in 100 trials, what is $\hat{p}$?