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Recovery of Distributions via Moments

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Abstract: The problem of recovering a cumulative distribution function (cdf) and corresponding density function from its moments is studied. This problem is a special case of the classical moment problem. The results obtained within the moment problem can be applied in many indirect models, e.g., those based on convolutions, mixtures, multiplicative censoring, and right-censoring, where the moments of unobserved distribution of actual interest can be easily estimated from the transformed moments of the observed distributions. Nonparametric estimation of a quantile function via moments of a target distribution represents another very interesting area where the moment problem arises. In all such models one can use the procedure, which recovers a function via its moments. In this article some properties of the proposed constructions are derived. The uniform rates of convergence of the approximants of cdf, its density function, and quantile function are obtained as well.

Contents

1. Introduction

The probabilistic Stielties moment problem can be described as follows: let a sequence $\nu = \{\mu_j, j = 0, 1, \dots\}$ of real numbers be given. Find a probability distribution on the non-negative real line $\mathbb{R}_+ = [0, \infty)$, such that $\mu_j = \int t^j d\,F(t)$ for $j \in \mathbb{N} = \{0, 1, \dots\}$. The classical Stieltjes moment problem was introduced first by Stieltjes [21]. When the support of the distribution F is a compact, say, $supp\{F\} = [0, T]$ with $T < \infty$, then the corresponding problem in known as a Hausdorff moment problem.

There are two important questions related to the Stieltjes (or Hausdorff) moment problem:

(i) If the distribution F exists, is it uniquely determined by the moments $\{\mu_j\}$?
(ii) How is this uniquely defined distribution F reconstructed?

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If there is a positive answer to the question (i) we say that a distribution F is moment-determinate (M-determinate), otherwise it is M-indeterminate.

In this paper we mainly address the question of recovering the M-determinate distribution (density and quantile functions) via its moments in the Hausdorff moment problem, i.e., we study the question (ii). Another question we focus on here is the estimation problem of unknown distribution and its quantile function, given the estimated moments of the target distribution.

It is known from the probabilistic moment problem that under suitable conditions M-determinate distribution is uniquely defined by its moments. There are many articles that investigated the conditions (for example, the Carlemann's and the Krein's conditions), under which the distributions are either M-determinate or M-indeterminate, see, Akhiezer [2], Feller [6], Lin [10-11], and Stoyanov [22-24] among others. However, there are very few works dealing with the reconstructions of distributions via their moments. Several inversion formulas were obtained by inverting the moment generating function and Laplace transform (Shohat and Tamarkin [20], Widder [27], Feller [6], Chauveau *et al.* [4], and Tagliani and Velasquez [25]). These methods are too restrictive, since there are many distributions for which the moment generating functions do not exist even though all the moments are finite.

The reconstruction of an M-determinate cdf by means of mixtures having the same assigned moments as the target distribution have been proposed in Lindsay *et al.* [12]. Note that this procedure requires calculations of high-order Hankel determinants, and due to ill-conditioning of the Hankel matrices this method is not useful when the number of assigned moments is large. The reconstruction of an unknown density functions using the Maximum Entropy principle with the specified ordinary and fractional moments has been studied in Kevasan and Kapur [9] and Novi Inverardi *et al.* [18], among others.

In Mnatsakanov and Ruymgaart [17] the constructions (2.2) and (3.13) (see Sections 2 and 3 below) have been introduced, and only their convergence in a weak sense have been established.

Different types of convergence of maximum entropy approximants have been studied by Borwein and Lewis [3], Frontini and Tagliani [7], and Novi Inverardi *et al.* [18], but the rates of approximations have not been established yet. Our construction enables us to derive the uniform rate of convergence for moment-recovered cdfs $F_{\alpha,\nu}$, corresponding quantile function approximate Q_α, and the uniform convergence of moment-recovered density approximant $f_{\alpha,\nu}$, as the parameter $\alpha \to \infty$. Another constructions of moment-recovered cdfs and pdfs (see, (3.13) and (3.14) in Remark 3.3) were proposed in Mnatsakanov [13-14], where the uniform and L_1-rates of approximations have been established for them.

The paper is organized as follows: in Section 2 we introduce some notations and assumptions, while in Section 3 we study the properties of $F_{\alpha,\nu}$ and $f_{\alpha,\nu}$. Note that our construction also gives a possibility to recover different distributions through the simple transformations of moment sequences of given distributions (Theorem 3.1 and Corollary 3.1). In Theorem 3.2 we state the uniform rate of convergence for moment-recovered cdfs. In Corollaries 3.2 and 3.3, and in Theorem 3.3 we applied the constructions (2.2) and (3.11) to recover pdf f, the quantile function Q, and corresponding quantile density function q of F given the moments of F. In Section 4 some other applications of the constructions (2.2) and (3.11) are discussed: the uniform convergence of the empirical counterparts of (2.2) and (3.11), the rate of approximation of moment-recovered quantile function (see (4.4) in Section 4) along with the demixing and deconvolution problems in several particular models.

Note that our approach is particularly applicable in situations where other estimators cannot be used, e.g., in situations where only moments (empirical) are available. The results obtained in this paper will not be compared with similar results derived by other methods. We only carry out the calculations of moment-recovered cdfs, pdfs, and quantile functions, and compare them with the target distributions via graphs in several simple examples. We also compare the performances of $F_{\alpha,\nu}$ and $f_{\alpha,\nu}$ with the similar constructions studied in Mnatsakanov [13-14] (see, Figures 1 (b) and 3 (b)). The moment-estimated quantile function $\hat{Q}_\alpha$ and well known Harrell-Davis quantile function estimator $\hat{Q}_{HD}$ (Sheater and Marron [19]) defined in (4.6) and (4.7), respectively, are compared as well (see, Figure 2 (b)).

2. Some Notations and Assumptions

Suppose that M-determinate cdf F is absolute continuous with respect to the Lebesgue measure and has support $[0,T]$, $T<\infty$. Denote the corresponding density function by f. Our method of recovering cdf $F(x)$, $0 \leqslant x \leqslant T$, is based on an inverse transformation that yields a solution of the Hausdorff moment problem.

Let us denote the ordinary moments of F by

$$\mu_{j,F} = \int t^j \, dF(t) = (\mathcal{K}F)(j), j \in \mathbb{N}, \tag{2.1}$$

and assume that the moment sequence $\nu = (\mu_{0,F},\, \mu_{1,F}, \dots)$ determines F uniquely.

An approximate inverse of the operator $\mathcal{K}$ from (2.1) constructed according to

$$\left(\mathcal{K}_\alpha^{-1}\nu\right)(x) = \sum_{k=0}^{[\alpha x]} \sum_{j=k}^{\infty} \frac{(-\alpha)^{j-k}}{(j-k)!}\, \frac{\alpha^k}{k!}\, \mu_{j,F},\ 0 \leqslant x \leqslant T\,,\ \alpha \in \mathbb{R}_+, \tag{2.2}$$

is such that $\mathcal{K}_\alpha^{-1}\,\mathcal{K}F \to_w F$, as $\alpha \to \infty$ (see, Mnatsakanov and Ruymgaart [17]). Here $\to_w$ denotes the weak convergence of cdfs, i.e. convergence at each continuity point of the limiting cdf.

The success of the inversion formula (2.2) hinges on the convergence

$$P_\alpha(t,x) = \sum_{k=0}^{[\alpha x]} \frac{(\alpha t)^k}{k!}\, e^{-\alpha t} \to \begin{cases} 1, & t < x \\ 0, & t > x \end{cases}\quad, \tag{2.3}$$

as $\alpha \to \infty$. This result is immediate from a suitable interpretation of the left hand side as a sum of Poisson probabilities.

For any moment sequence $\nu = \{\nu_j, j \in \mathbb{N}\}$, let us denote by F_ν the cdf recovered via $F_{\alpha,\nu} = \mathcal{K}_\alpha^{-1}\nu$ according to (2.2), when $\alpha \to \infty$, i.e.

$$F_{\alpha,\nu} \to_w F_\nu\,,\ as\ \alpha \to \infty\,. \tag{2.4}$$

Note that if $\nu = \{\mu_{j,F}, j \in \mathbb{N}\}$ is the moment sequence of F, the statement (2.4) with $F_\nu = F$ is proved in Mnatsakanov and Ruymgaart [17].

To recover a pdf f via its moment sequence $\{\mu_{j,F}, j \in \mathbb{N}\}$, consider the ratio:

$$f_{\alpha,\nu}(x) = \frac{\Delta F_{\alpha,\nu}(x)}{\Delta},\ \ \Delta = \frac{1}{\alpha}\,, \tag{2.5}$$

where $\Delta F_{\alpha,\nu}(x) = F_{\alpha,\nu}(x+\Delta) - F_{\alpha,\nu}(x)$ and $\alpha \to \infty$.

In the sequel the uniform convergence on any bounded interval from $\mathbb{R}_+$ will be denoted by $\longrightarrow_u$, while the L_1-norm between two functions f_1 and f_2 by $|| f_1 - f_2 ||_{L_1}$. Note also that the statements from Sections 3 and 4 are valid for distributions defined on any compact $[0, T], T < \infty$. Without loss of generality we assume that F has support $[0, 1]$.

3. Asymptotic Properties of $F_{\alpha,\nu}$ and $f_{\alpha,\nu}$

In this Section we present some asymptotic properties of moment-recovered cdf $F_{\alpha,\nu}$ and moment-recovered pdf $f_{\alpha,\nu}$ functions based on transformation $\mathcal{K}_\alpha^{-1}\nu$ from (2.2). The uniform approximation rate of $F_{\alpha,\nu}$ and the uniform convergence of $f_{\alpha,\nu}$ are derived as well.

Denote the family of all cdfs defined on $[0, 1]$ by $\mathbb{F}$. The construction (2.2) gives us the possibility to recover also two non-linear operators $\mathcal{A}_k : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, $k = 1, 2$, defined as follows: denote the convolution with respect to the multiplication operation on $\mathbb{R}_+$ by

$$F_1 \otimes F_2(x) = \int F_1(x/\tau)\, dF_2(\tau) := \mathcal{A}_1(F_1, F_2)(x), \quad 0 \leqslant x \leqslant 1, \tag{3.1}$$

while the convolution with respect to the addition operation is denoted by

$$F_1 \star F_2(x) = \int F_1(x - \tau)\, dF_2(\tau) := \mathcal{A}_2(F_1, F_2)(x), \quad 0 \leqslant x \leqslant 2\,.$$

For any two moment sequences $\nu_1 = \{\mu_{j,F_1}, j \in \mathbb{N}\}$ and $\nu_2 = \{\mu_{j,F_2}, j \in \mathbb{N}\}$, let us use the following notations: $\nu_1 \odot \nu_2 = \{\mu_{j,F_1} \times \mu_{j,F_2}, j \in \mathbb{N}\}$ and $\nu_1 \oplus \nu_2 = \{\bar{\nu}_j, j \in \mathbb{N}\}$ where

$$\bar{\nu}_j = \sum_{m=0}^{j} \binom{j}{m} \mu_{m,F_1} \times \mu_{j-m,F_2}\,. \tag{3.2}$$

Also denote by $F \circ \phi^{-1}$ the composition $F(\phi^{-1}(x)), x \in [0, 1]$, with ϕ - continuous and increasing function $\phi : [0, 1] \to [0, 1]$.

Since cdfs $\mathcal{A}_1(F_1, F_2) = F_1 \otimes F_2$, $\mathcal{A}_2(F_1, F_2) = F_1 \star F_2$, and $F \circ \phi^{-1}$ have a compact support, they all are M-determinate and have the moment sequences $\nu_1 \odot \nu_2$, $\nu_1 \oplus \nu_2$, and $\nu = \{\bar{\mu}_j\,, j \in \mathbb{N}\}$, with

$$\bar{\mu}_j = \int [\phi(t)]^j\, dF(t), \tag{3.3}$$

respectively. Hence, applying the Theorem 3.1 from Mnatsakanov and Ruymgaart [17] we derive

Theorem 3.1.
(i) *If* $\nu = \nu_1 \odot \nu_2$, *then (2.4) holds with* $F_\nu = F_1 \otimes F_2$;

(ii) *If* $\nu = \nu_1 \oplus \nu_2$, *then (2.4) holds with* $F_\nu = F_1 \star F_2$;

(iii) *If* $\nu = \{\bar{\mu}_j\,, j \in \mathbb{N}\}$, *with* $\bar{\mu}_j$ *defined in (3.3), then (2.4) holds with* $F_\nu = F \circ \phi^{-1}$.

In the following statement let us use the notations: $\mu_F^{\odot k} = \{\mu_{j,F}^k, j \in \mathbb{N}\}$ and $F^{\otimes k} = F \otimes \cdots \otimes F$ for corresponding k-fold convolution (cf. (3.1)).

COROLLARY 3.1.
(i) If $\nu = \{\mu_{aj+b,F}/\mu_{b,F}, j \in \mathbb{N}\}$ *for some* $a > 0$ *and* $b \geqslant 0$, *then (2.4) holds with*

$$F_\nu(x) = \frac{1}{\mu_{b,F}} \int_0^{x^{1/a}} t^b \, dF(t)\,; \tag{3.4}$$

in particular, if $\nu = \{\mu_{aj,F}, j \in \mathbb{N}\}$, *then* $F_\nu(x) = F(x^{1/a})$ *in (3.4);*

(ii) If $\nu = \beta_1\nu_1 + \beta_2\nu_2$ *with* $\beta_1 + \beta_2 = 1$, $\beta_1, \beta_2 > 0$, *then (2.4) holds with* $F_\nu = \beta_1 F_1 + \beta_1 F_2$;

(iii) If $\nu = \{a^j \mu_{j,F}, j \in \mathbb{N}\}$, *then (2.4) holds with* $F_\nu(x) = F(x/a)$;

(iv) If $\nu = \sum_{k=1}^m \beta_k \mu_F^{\odot k}$, *where* $\sum_{k=1}^m \beta_k = 1, \beta_k > 0$, *then (2.4) holds with*

$$F_\nu = \sum_{k=1}^m \beta_k F^{\otimes k}\,.$$

Proof. The statements (i) can be justified in a similar way we proceeded in Theorem 3.1. The result of (ii) is a consequence of the linearity of $\mathcal{K}_\alpha^{-1}\,\nu$, while, the statement (iii) is a special case of Theorem 3.1 (i), since the distribution with the moments $\mu_j = a^j, j \in \mathbb{N}$, is degenerated at a. The proof of (iv) follows from Theorem 3.1 (i) and Corollary 3.1 (ii).

REMARK 3.1. If $a = 1$ in Corollary 3.1*(i)*, then cdf F_ν from (3.5) represents the biased sampling model with the weight function $w(t) = t^b$.

The construction (2.2) is also useful in the problem of recovering the quantile function $Q(t) = \inf\{x : F(x) \geqslant t\}$ via moments (see (4.5) in Section 4). Let us denote $Q_\alpha = F_{\alpha,\nu_Q}$, where

$$\nu_Q = \{\int_0^1 [F(u)]^j \, du, j \in \mathbb{N}\}\,. \tag{3.5}$$

The following statement is true:

COROLLARY 3.2. *If* F *is continuous, then* $Q_\alpha \to_w Q$, *as* $\alpha \to \infty$.

Proof. Replacing the functions ϕ and F in (3.3) by F and uniform cdf defined on $[0, 1]$, respectively, we obtain from Theorem 3.1 (iii) that $Q_\alpha = F_{\alpha,\nu_Q} \to_w F_\nu = F^{-1}$ as $\alpha \to \infty$.

Under additional condition on the smoothness of F one can obtain the uniform rate of convergence in (2.4) and, hence, in Theorem 3.1 too. Consider the following condition

$$F'' = f' \text{ is bounded on } [0, 1]\,. \tag{3.6}$$

Theorem 3.2. *If $\nu = \{\mu_{j,F}, j \in \mathbb{N}\}$, then under the condition (3.6), we have*

$$\sup_{0\leqslant x\leqslant 1} \left|F_{\alpha,\nu}(x) - F(x)\right| = O\left(\frac{1}{\alpha}\right), \quad \text{as } \alpha \to \infty. \tag{3.7}$$

Proof. Let us use the following representation

$$P_\alpha(t,x) = \mathbf{P}\{N_{\alpha t} \leqslant \alpha x\} = \mathbf{P}\{S_{[\alpha x]} \geqslant \alpha t\}.$$

Here $\{N_{\alpha t}, t \in [0,1]\}$ is a Poisson process with intensity αt, $S_m = \sum_{k=0}^m \xi_k, S_0 = 0$, with ξ_k being *iid* $Exp(1)$ random variables. To show (3.7), let us use the integration by parts in

$$\begin{aligned} F_{\alpha,\nu}(x) = (\mathcal{K}_\alpha^{-1}\,\nu)(x) &= \int_0^1 \sum_{k=0}^{[\alpha x]} \frac{(\alpha t)^k}{k!} \sum_{j=k}^{\infty} \frac{(-\alpha t)^{j-k}}{(j-k)!}\, dF(t) \\ &= \int_0^1 P_\alpha(t,x) d\,F(t) = \int_0^1 \mathbf{P}\{S_{[\alpha x]} \geqslant \alpha t\} dF(t) \\ &= F(t)\,\mathbf{P}\{S_{[\alpha x]} \geqslant \alpha t\}\Big|_0^1 - \int_0^1 F(t)\, d\,\mathbf{P}\{S_{[\alpha x]} \geqslant \alpha t\} \\ &= \mathbf{P}\{S_{[\alpha x]} \geqslant \alpha\} + \int_0^1 F(t)\, d\,\mathbf{P}\{S_{[\alpha x]} \leqslant \alpha t\} = \int_0^\infty F(t)\, d\,\mathbf{P}\{S_{[\alpha x]} \leqslant \alpha t\}. \end{aligned} \tag{3.8}$$

So that the condition (3.6) and the argument used in Adell and de la Cal [1] yield the proof of the Theorem 3.2.

Remark 3.2. Assume $supp\{F\} = \mathbb{R}_+$. In this case $F_{\alpha,\nu}(x) = \int_0^\infty P_\alpha(t,x) d\,F(t)$ (cf. with (3.8)). According to Mnatsakanov and Klaassen [16] (see the proof of Theorem 3.1), one can derive the exact rate of approximation of $F_{\alpha,\nu}$ in the space $L_2(\mathbb{R}_+, dF)$. Namely, under the condition that pdf f is bounded, say by $C > 0$, we have

$$\int_0^\infty \Big(F_{\alpha,\nu}(x) - F(x)\Big)^2 dF(x) \leq \frac{2\,C}{\alpha}\,.$$

Now let us consider the moment-recovered density functions $f_{\alpha,\nu}$ defined in (2.5) and denote by $\Delta(f,\delta) = \sup_{|t-s|\leq\delta} |f(t) - f(s)|$ the modulus of continuity of f, where $0 < \delta < 1$. The following statement is true:

Theorem 3.3. *If pdf f is continuous on* $[0,\,1]$, *then $f_{\alpha,\nu} \longrightarrow_u f$ and*

$$||\, f_{\alpha,\nu} - f\,|| \leq \Delta(f,\delta) + \frac{2\,||\,f\,||}{\alpha\delta^2} + o\Big(\frac{1}{\alpha}\Big), \quad as \;\; \alpha \to \infty. \tag{3.9}$$

Proof. Let us derive only (3.9). Since $[\alpha(x + 1/\alpha)] = [\alpha x] + 1$, for any $x \in [0,1]$, we have

$$f_{\alpha,\nu}(x) = \alpha\,\Big[\sum_{k=0}^{[\alpha x]+1} \sum_{j=k}^{\infty} \frac{(-\alpha)^{j-k}}{(j-k)!}\,\frac{\alpha^k}{k!}\,\mu_{j,F} - \sum_{k=0}^{[\alpha x]}\sum_{j=k}^{\infty} \frac{(-\alpha)^{j-k}}{(j-k)!}\,\frac{\alpha^k}{k!}\,\mu_{j,F}\Big], \tag{3.10}$$

and, after a simple algebra (3.10) yields

$$f_{\alpha,\nu}(x) = \frac{\alpha^{[\alpha x]+2}}{\Gamma([\alpha x]+2)} \cdot \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{m!}\,\mu_{m+[\alpha x]+1,F}\,. \tag{3.11}$$

Let denote by

$$g(t,a,b) = \frac{b^a\, t^{a-1}}{\Gamma(a)}\, e^{-b\,t}\ ,\ t > 0\ ,\quad \text{for any } a\,,\, b\, >\, 0\,,$$

a gamma pdf with the shape and the rate parameters a and b, respectively. Substitution (2.1) into the right hand side of (3.11) gives

$$\text{(3.12)}\qquad f_{\alpha,\nu}(x) = \frac{\alpha^{[\alpha x]+2}}{\Gamma([\alpha x]+2)} \int_0^1 \sum_{m=0}^{\infty} \frac{(-\alpha t)^m}{m!} t^{[\alpha x]+1}\, dF(t)$$
$$= \int_0^1 g(t,[\alpha x]+2,\alpha)\, f(t) dt.$$

To show (3.9), let us note that pdf g in (3.12) has the mean $([\alpha x]+2)/\alpha$ and the variance $([\alpha x]+2)/\alpha^2$, respectively. Application of the Tchebishev's inequality and splitting the integral on the right hand side of (3.12) into two parts yields

$$\sup_{0\le x\le 1} \left(\int_{|t-x|\le\delta} + \int_{|t-x|>\delta} \right) |f(t)-f(x)\,|\, g(t,[\alpha x]+2,\alpha)\, dt$$
$$\le \Delta(f,\delta) + 2\,||\,f\,||\, \sup_{0\le x\le 1} \int_{|t-x|>\delta} g(t,[\alpha x]+2,\alpha)\, dt$$
$$\le \Delta(f,\delta) + \frac{2\,||\,f\,||}{\alpha\delta^2} + o\Big(\frac{1}{\alpha}\Big)\ ,$$

as $\alpha \to \infty$ and $0 < \delta < 1$.

Remark 3.3. In Mnatsakanov [13-14] the uniform and L_1-rates of moment-recovered approximations of F and f defined accordingly by

$$\text{(3.13)}\qquad F^*_{\alpha,\nu}(x) = \sum_{k=0}^{[\alpha x]} \sum_{j=k}^{\alpha} \binom{\alpha}{j} \binom{j}{k} (-1)^{j-k}\, \mu_{j,F}$$

and

$$\text{(3.14)}\qquad f^*{}_{\alpha,\nu}(x) = \frac{\Gamma(\alpha+2)}{\Gamma([\alpha x]+1)} \sum_{m=0}^{\alpha-[\alpha x]} \frac{(-1)^m\, \mu_{m+[\alpha x],F}}{m!\,(\alpha-[\alpha x]-m)!},\ \ x\in[0,1]\,,\ \alpha\in\mathbb{N},$$

are established. In Section 4, see Example 4.2, the cdf $F(x) = x^3 - 3\,x^3 \ln x$ and its density function $f(x) = -9x^2 \ln x, 0 \leqslant t \leqslant 1$, are recovered using both $F_{\alpha,\nu}$ and $F^*{}_{\alpha,\nu}$, and $f_{\alpha,\nu}$ and $f^*{}_{\alpha,\nu}$ constructions, (see Figures 1 (b) and 3 (b), respectively).

The formulas (3.11) and (3.14) with $\nu = \nu_Q$ defined according to (3.5) can be used to recover a quantile density function

$$q(x) = Q'(x) = \frac{1}{f(F^{-1}(x))}\,,\qquad x\in[0,1].$$

For example, consider $f_{\alpha,\nu_Q} := q_\alpha$: the application of the first line in (3.12) with F^{-1} instead of F yields

$$\text{(3.15)}\qquad q_\alpha(x) = \int_0^1 g(F(u),[\alpha x]+2,\alpha)\, du$$

and corresponding moment-recovered quantile density function

$$q_{\alpha,\beta}(x) = \int_0^1 g(F_{\beta,\nu}(u), [\alpha x] + 2, \alpha)\, du\,, \quad \alpha\,, \beta \in \mathbb{N}.$$

Here $F_{\beta,\nu}$ is a moment-recovered cdf of F. As a consequence of Theorem 3.3 we have the following

COROLLARY 3.3. *If $q_\alpha = f_{\alpha,\nu_Q}$, with ν_Q defined in (3.5), and f is continuous on $[0,\,1]$ with $\inf_{0\leqslant x\leqslant 1} f(x) > \gamma > 0$, then $q_\alpha \longrightarrow_u q$ and*

$$(3.16)\qquad || \, q_\alpha - q \, || \leq \frac{\Delta(f,\delta)}{\gamma^2} + \frac{2\,||\,f\,||}{\alpha\,\delta^2\,\gamma^2} + o\Big(\frac{1}{\alpha}\Big)\,, \quad \textit{as} \ \ \alpha \to \infty.$$

Finally, note that taking $\nu = \nu_Q$ in (3.14), we derive another approximation $q^*_\alpha = f^*_{\alpha,\nu_Q}$ of q based on Beta densities $\beta(\cdot, a, b)$ with the shape parameters $a = [\alpha x] + 1$ and $b = \alpha - [\alpha x] + 1$:

$$(3.17)\qquad q^*_\alpha(x) = \int_0^1 \beta(F(u), [\alpha x] + 1, \alpha - [\alpha x] + 1)\, du\,.$$

4. Some Applications and Examples

In this Section the construction of the moment-recovered cdf $F_{\alpha,\nu}$ is applied to the problem of nonparametric estimation of a cdf, its density and a quantile functions as well as to the problem of demixing in exponential, binomial and negative binomial mixtures, and deconvolution in error-in-variable model. In Theorems 4.1 we derive the uniform rate of convergence for the empirical counterpart of $F_{\alpha,\nu}$ denoted by $\widetilde{F}_\alpha$, i.e. for $\widetilde{F}_\alpha = F_{\alpha,\hat{\nu}}$, where $\hat{\nu}$ being the sequence of all empirical moments of the sample from F. In Theorem 4.2 we establish the L_1-consistency of the density estimate when the sample $X_1, \ldots, X_n$ from F is drawn. In Theorem 4.3 the uniform rate of approximation for moment-recovered quantile function of F is obtained. Finally, the graphs of moment-recovered cdfs, pdfs, and quantile functions are presented in Figures 1-3.

Direct model. Assume we have n copies $X_1, \ldots, X_n$ from cdf F defined on $[0, 1]$. Denote by $\hat{F}_n$ the ordinary empirical cdf (ecdf) of the sample $X_1, \ldots, X_n$:

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^n I_{[0,t]}(X_i)\,, \quad 0 \leqslant t \leqslant 1\,.$$

Substitution of the empirical moments

$$\hat{\nu}_j = \frac{1}{n}\sum_{i=1}^n X_i^j\,,\, j \in \mathbb{N}\,,$$

instead of $\mu_{j,F}$ into (2.2) yields to

$$\widetilde{F}_\alpha(x) = F_{\alpha,\hat{\nu}}(x) = \int_0^1 P_\alpha(t,x) d\,\hat{F}_n(t) = \int_0^1 \mathbf{P}\{S_{[\alpha x]} \geqslant \alpha t\} d\,\hat{F}_n(t)\,.$$

Furthermore, the empirical analogue of (3.8) admits a similar representation

$$\widetilde{F}_\alpha(x) = \int_0^\infty \hat{F}_n(t)\, d\,\mathbf{P}\{S_{[\alpha x]} \leqslant \alpha t\}.$$

The application of the Theorem 3.2 and the asymptotic properties of $\hat{F}_n$ yield

THEOREM 4.1. *If* $\nu = \{\mu_{j,F}, j \in \mathbb{N}\}$, *then under the condition (3.6) we have*

$$\sup_{0\leqslant x\leqslant 1} \left|\widetilde{F}_\alpha(x) - F(x)\right| = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{1}{\alpha}\right) \quad \text{a.s.}, \quad \text{as} \quad \alpha\,, n \to \infty. \tag{4.1}$$

REMARK 4.1. In Mnatsakanov and Ruymgaart [17] the weak convergence of the moment-empirical processes $\{\sqrt{n}\{\widetilde{F}_n(t) - F(t)\},\ t \in [0,1]\}$ to the Brownian bridge is obtained.

Of course, when the sample is directly drawn from the cdf F of actual interest, one might use the ordinary ecdf $\hat{F}_n$ and empirical process $U_n = \sqrt{n}\,(\hat{F}_n - F)$. The result mentioned in Remark 4.1 yields, that even if the only information available is the empirical moments, we still can construct different test statistics based on the moment-empirical processes $\widetilde{U}_n = \sqrt{n}\,(\widetilde{F}_n - F)$.

On the other hand, using the construction (3.11), one can estimate the density function f given only the estimated or empirical moments in:

$$f_{\alpha,\hat{\nu}}(x) = \frac{\alpha^{[\alpha x]+2}}{\Gamma([\alpha x]+2)} \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{m!}\,\hat{\nu}_{m+[\alpha x]+1}\,,\ x \in [0,1]\,. \tag{4.2}$$

REMARK 4.2. In practice, the parameter α as well as the number of summands in (4.2) (and the number of summands in the inner summation of $F_{\alpha,\hat{\nu}}$) can be chosen as the functions of n: $\alpha = \alpha(n) \to \infty$ and $M = M(n) \to \infty$ as $n \to \infty$, that optimize the accuracy of corresponding estimates. Further analysis is required to derive the asymptotic forms of $\alpha(n)$ and $M(n)$ as $n \to \infty$. This question is currently under investigation and is beyond the scope of the present article.

Note that when the sample from F is given, the construction (4.2) yields the estimate $\hat{f}_\alpha(x) = f_{\alpha,\hat{\nu}}$ with $\hat{\nu} = \{\hat{\nu}_j, j \in \mathbb{N}\}$:

$$\hat{f}_\alpha(x) = \frac{\alpha}{n}\sum_{i=1}^{n} \frac{(\alpha X_i)^{[\alpha x]+1}}{([\alpha x]+1)!}\,e^{-\alpha X_i} = \frac{1}{n}\sum_{i=1}^{n} g(X_i, [\alpha x]+2, \alpha),\ \ x \in [0,1]\,.$$

Here $g(\cdot, [\alpha x]+2, \alpha)$ defined in (3.12). The estimator $\hat{f}_\alpha$ does not represent a traditional kernel density estimator of f. It is defined by a δ-sequence, which consists of the gamma density functions of varying shapes (the shape and the rate parameters are equal to $[\alpha x] + 2$ and α, respectively). It is natural to use this estimate when $supp\{F\} = [0, \infty)$, since, in this case, the supports of f and gamma kernel densities coincide and one avoids the boundary effect of $\hat{f}_\alpha$ (cf. Chen [5]).

Some asymptotic properties such as the convergence in probability of $\hat{f}_\alpha$ uniformly on any bounded interval and the Integrated Mean Squared Error (*IMSE*) of $\hat{f}_\alpha$ have been studied in Mnatsakanov and Ruymgaart [17] and Chen [5], respectively.

Applying the results from Mnatsakanov and Khmaladze [15], where the necessary and sufficient conditions for L_1-consistency of general kernel density estimates are established, one can prove in a similar way (cf. Mnatsakanov [14]) the following statement:

THEOREM 4.2. *If f is continuous on* $[0,1]$, *then*

$$E\,||\,\hat{f}_\alpha - f\,||_{L_1} \to 0\,,\quad as\quad \frac{\sqrt{\alpha}}{n} \to 0\ \ and\ \ \alpha\,,\, n\ \to \infty.$$

Note here only that the proof of Theorem 4.2 is based on the result of Theorem 1 from Mnatsakanov and Khmaladze [15] combined with the statement of Theorem 3.3 that yields $\| f_{\alpha,\nu} - f \|_{L_1} \to 0$ (where $f_{\alpha,\nu}(x) = E\hat{f}_\alpha(x)$). Here we also use the inequality

$$g(t, [\alpha x] + 2, \alpha) \le \frac{\sqrt{\alpha}}{\sqrt{2\pi\, x}}\,, \;\; 0 < x < 1\,,$$

that follows from the Stirling's formula for $\Gamma([\alpha x] + 2)$ and the property that the function $g(\cdot, [\alpha x] + 2, \alpha)$ defined in (3.12) attends its maximum at $t_{max} = ([\alpha x] + 1)/\alpha$.

Exponential mixture model. Suppose, one observes a random sample $Y_1, \dots, Y_n$ from the mixture of exponentials

$$G(x) = \int_0^T (1 - e^{-x/\tau})\, dF(\tau)\,, \;\; x \geqslant 0\,.$$

We arrive at convolution (3.1) with $G = Exp(1) \otimes F$. The unknown cdf F can be recovered according to the construction $F_{\alpha,\nu} = \mathcal{K}_\alpha^{-1}\,\nu$ with $\nu = \{\mu_{j,G}/j!, j \in \mathbb{N}\}$. Similarly, given the sample $Y_1, \dots, Y_n$ from G and taking $F_{\alpha,\hat{\nu}} = \mathcal{K}_\alpha^{-1}\,\hat{\nu}$, where $\hat{\nu} = \{\hat{\mu}_{j,G}/j!, j \in \mathbb{N}\}$, we obtain the estimate of F. Here $\{\hat{\mu}_{j,G}, j \in \mathbb{N}\}$ are the empirical moments of the sample $Y_1, \dots, Y_n$. The regularized inversion of the noisy Laplace transform and the L_2-rate of convergence were obtained in Chauveau *et al.* [4].

Binomial and negative binomial mixture models. Suppose, one observes a random sample $Y_1, \dots, Y_n$ from the binomial or negative binomial mixture distributions, respectively:

$$p(x) := P(Y = x) = \int_0^1 \binom{m}{x} \tau^x\, (1-\tau)^{m-x} dF(\tau)\,, \;\; x = 0, \dots m\,,$$

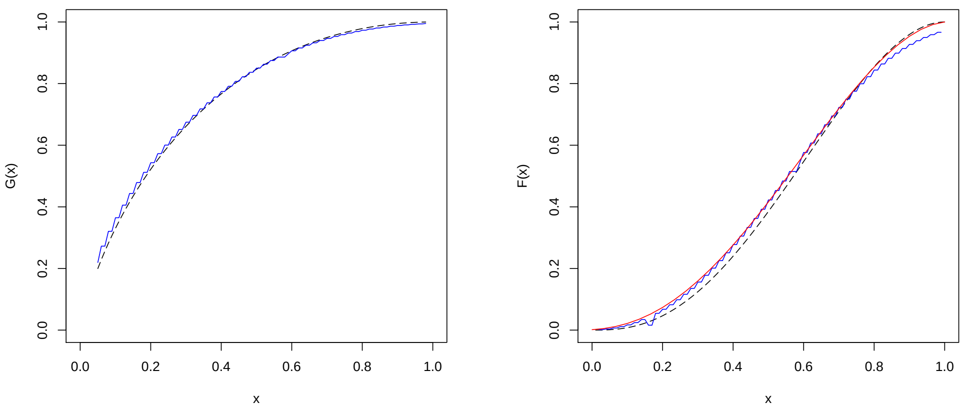
$$p(x) := P(Y = x) = \int_0^1 \frac{\Gamma(r + x)}{\Gamma(r)\, x!} \left(\frac{1}{1+\tau}\right)^r \left(\frac{\tau}{1+\tau}\right)^x dG(\tau)\,, \;\; x = 0, 1, \dots\,,$$

where m and r are given positive integers. Assume that the unknown mixing cdfs F and G are such that F has at most $\frac{m+1}{2}$ support points in $(0, 1)$, while G is a right continuous cdf on $(0, 1)$. In both models the mixing distributions are identifiable (see, for example, Teicher [26] for binomial mixture model). Note also that the jth moments of F and G are related to the jth factorial moments of corresponding Y_i's in the following ways:

$$\mu_{j,F} = \frac{1}{m^{[j]}}\, E(Y_1^{[j]}) \;\; \text{and} \;\; \mu_{j,G} = \frac{1}{r_{(j)}}\, E(Y_1^{[j]})\,.$$

Here $y^{[j]} = y(y-1)\cdots(y-j+1)$ and $r_{(j)} = r(r+1)\cdots(r+j-1)$. To estimate F and G one can use the moment-recovered formulas (2.2) or (3.13) with $\mu_{j,F}$ and $\mu_{j,G}$ defined in previous two equations where the theoretical factorial moments are replaced by corresponding empirical counterparts. The asymptotic properties of the derived estimators of F and G will be studied in a separate work.

Deconvolution problem: error-in-variable model. Assume now we observe a random variable $Y = X + U$, with cdf G, where U (the error) has some known symmetric distribution F_2, X has a cdf F_1 with a support $[0, T]$, and U and X are



(a) (b)

FIG 1. *(a) Approximation of* $G(x) = x - x \ln x$ *by* $F_{\alpha,\nu}$ *and (b) Approximation of* $G(x^3)$ *by* $F_{\alpha,\nu}$ *and by* $F^*_{\alpha,\nu}$

independent. This model, known as an error-in-variable model, corresponds to the convolution $G = F_1 \star F_2$. Assuming that all moments of X and U exist, the moments $\{\bar{\nu}_j, j \in \mathbb{N}\}$ of Y are described according to (3.2). Hence, given the moments of U (with $E(U) = 0$), we can recalculate the moments of F_1 as follows: $\mu_{1,F_1} = \bar{\nu}_1$, $\mu_{2,F_1} = \bar{\nu}_2 - \mu_{2,F_2}$, and so on. So that, assuming that we already calculated μ_{k,F_1}, or estimated them by μ^*_{k,F_1} for $1 \leqslant k \leqslant j-2$, we will have, for any $j \geqslant 1$:

$$\mu_{j,F_1} = \bar{\nu}_j - \sum_{m=2}^{j} \binom{j}{m} \mu_{m,F_2} \times \mu_{j-m,F_1}$$

or, respectively,

$$\mu^*_{j,F_1} = \hat{\mu}_{j,G} - \sum_{m=2}^{j} \binom{j}{m} \mu_{m,F_2} \times \mu^*_{j-m,F_1}$$

given the sample $Y_1, \ldots, Y_n$ from cdf G. Now the moment-recovered estimate of F_1 will have the form $F_{\alpha,\hat{\nu}} = \mathcal{K}_\alpha^{-1}\,\hat{\nu}$, where $\hat{\nu} = \{\mu^*_{j,F_1}, j \in \mathbb{N}\}$. The alternative construction of the kernel type estimate of F_1 based on the Fourier transforms is studied in Hall and Lahiri [8], where the $\sqrt{n}$-consistency and other properties of the estimated moments $\mu^*_{j,F_1}, j \in \mathbb{N}$, are derived as well.

Example 4.1. Consider the moment sequence $\mu = \{1/(j+1), j \in \mathbb{N}\}$. The corresponding moment-recovered distribution $F_{\alpha,\mu} = \mathcal{K}_\alpha^{-1}\,\mu$ is a good approximation of $F(x) = x$ already with $\alpha = 50$ and $M = 100$.

Assume now that we want to recover the distribution G with corresponding moments $\nu_{j,G} = 1/(j+1)^2, j \in \mathbb{N}$. Since we can represent $\nu_G = \mu \odot \mu$, we conclude from Theorem 3.1, see (i), that $G = F \otimes F$, with $F(x) = x$, and hence $G(x) = x - x \ln x, 0 \leqslant x \leqslant 1$. We plotted the curves of F_{α,ν_G} (the solid line) and G (the dashed line) on Figure 1 (a). We took $\alpha = 50$ and $M = 200$, the number of terms in the inner summation of the formula (2.2). From Figure 1 (a) we can see that the approximation of G by F_{α,ν_G} at $x = 0$ is not as good as inside of the interval $[0, 1]$. This happened because the condition (3.6) from Theorem 3.2 is not valid for function $g'(x) = G''(x) = -1/x$.

Example 4.2. To recover the distribution F via moments $\nu_j = 9/(j+3)^2, j \in \mathbb{N}$, note that $\nu_j = \nu_{aj,G}$, with $a = 1/3$. Hence, $F(x) = G(x^3) = x^3 - x^3 \ln(x^3), 0 \leqslant x \leqslant 1$ (Corollary 3.1, see (i)). We conducted computations of moment-recovered cdf $F_{\alpha,\nu}$ when $\alpha = 50$ and the number of terms in the inner summation of the formula (2.2) is equal to 200. Also, we calculated $F^*_{\alpha,\nu}$ defined in (3.13) with $\alpha = 32$. See Figure 1 (b), where we plotted $F_{\alpha,\nu}$ (the solid blue line), $F^*_{\alpha,\nu}$ (the solid red line), and F (the dashed line), respectively. These two approximations of cdf F justify a good fit already with $\alpha = 50$ and $M = 200$ for the first one and with $\alpha = 32$ for the second one. From Figure 1 (b) we can see that the performance of $F^*_{\alpha,\nu}$ is slightly better compared to $F_{\alpha,\nu}$: $F^*_{\alpha,\nu}$ does not have the "boundary" effect around $x = 1$.

Estimation of a quantile function Q and quantile density function q. Assume that a random variable X has a continuous cdf F defined on $[0, 1]$. To approximate (estimate) the quantile function Q given only the moments (estimated moments) of F, one can use the Corollary 3.2. Indeed, after a simple algebra, we have

$$Q_\alpha(x) = F_{\alpha,\nu_Q}(x) = \int_0^1 P_\alpha\big(F(u), x\big)\, du\,, \quad 0 \leqslant x \leqslant 1\,, \tag{4.3}$$

where ν_Q and $P_\alpha(\cdot,\cdot)$ are defined in (3.5) and in (2.3), respectively. Comparing (4.3) and (3.8) we can prove in a similar way (see, the proof of Theorem 3.2) the following

THEOREM 4.3. *If f' is bounded and $\inf_{0\leqslant x\leqslant 1} f(x) > \gamma > 0$, then*

$$\sup_{0\leqslant x\leqslant 1} \big|Q_\alpha(x) - Q(x)\big| = O\left(\frac{1}{\alpha}\right), \quad \text{as } \alpha \to \infty. \tag{4.4}$$

Now, given only the moment sequence ν of F, one can construct the approximation $Q_{\alpha,\beta}$ of Q by substituting the moment-recovered cdf $F_{\beta,\nu}$ (instead of F) in the right hand side of (4.3). Let us denote the corresponding approximant of Q by

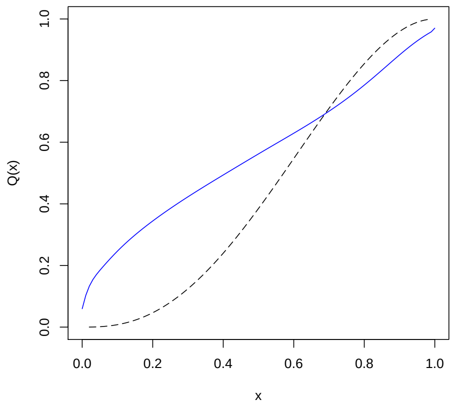
$$Q_{\alpha,\beta}(x) = \int_0^1 P_\alpha\big(F_{\beta,\nu}(u), x\big)\, du\,, \quad \alpha\ , \beta \in \mathbb{N}. \tag{4.5}$$

On Figure 2 (a) we plotted cdf $F(x) = x^3 - x^3 \ln(x^3)$ (the dashed line), introduced in Example 4.2, and its quantile approximation $Q_{\alpha,\beta}$ (the solid line), when $\nu = \{9/(j+3)^2, j \in \mathbb{N}\}$, $\alpha = \beta = 100$, and $M = 200$.

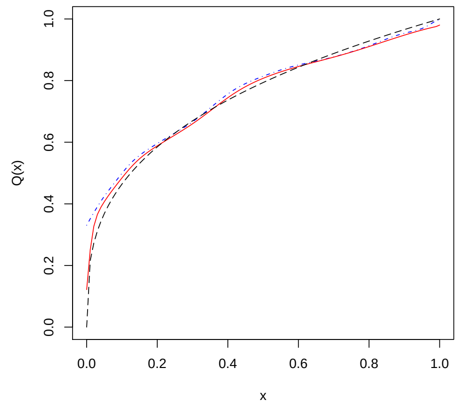
Now, let us use the empirical cdf $\hat{F}_n$ instead of F in (4.3), (3.15), and in (3.17), when the distribution F is observed via the sample $X_1, \ldots, X_n$. These yield the following estimators, respectively, based on the spacings $\Delta X_{(i)} = X_{(i)} - X_{(i-1)}, i = 1, \ldots, n+1$:

$$\hat{Q}_\alpha(x) = F_{\alpha,\hat{\nu}_Q}(x) = \int_0^1 P_\alpha\big(\hat{F}_n(u), x\big)\, du = \sum_{i=1}^{n+1} \Delta X_{(i)}\, P_\alpha\big(\frac{i-1}{n}, x\big)\,, \tag{4.6}$$

$$\hat{q}_\alpha(x) = \int_0^1 g(\hat{F}_n(u), [\alpha x] + 2, \alpha)\, du = \sum_{i=1}^{n+1} \Delta X_{(i)}\, g\big(\frac{i-1}{n}, [\alpha x] + 2, \alpha\big)\,,$$



(a)



(b)

FIG 2. *(a) Approximation of Q by $Q_{\alpha,\beta}$ and (b) Estimation of $Q(x) = x^{1/3}$ by $\hat{Q}_\alpha$ and by $\hat{Q}_{HD}$*

and

$$\hat{q}^*_\alpha(x) = \sum_{i=1}^{n+1} \Delta X_{(i)}\, \beta\Big(\frac{i-1}{n}, [\alpha x]+1, \alpha - [\alpha x] + 1\Big)\,.$$

Here $\hat{\nu}_Q = \{\int_0^1 [\hat{F}_n(u)]^j du, j \in \mathbb{N}\}$, while $X_{(i)}, i = 1, \ldots, n, X_{(0)} = 0, X_{(n+1)} = 1$, are the order statistics of the sample $X_1, \ldots, X_n$.

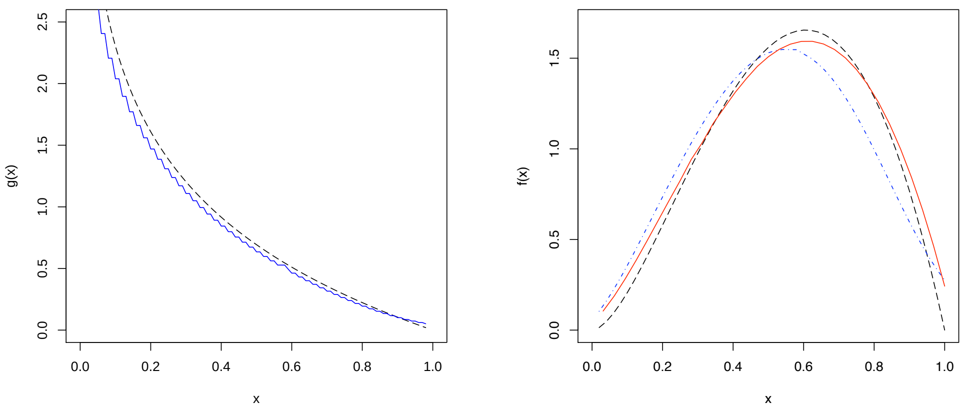
Now, let us compare the curves of $\hat{Q}_\alpha$ and well known Harrell-Davis estimator

$$\hat{Q}_{HD}(x) = \sum_{i=1}^{n} X_{(i)}\, \Delta Beta\left(\frac{i}{n},\, (n+1)x,\, (n+1)(1-x)\right), \tag{4.7}$$

where $Beta\,(\cdot, a, b)$ denotes the cdf of a $Beta$ distribution with the shape parameters $a > 0$ and $b > 0$. For asymptotic expressions of MSE and the bias term of $\hat{Q}_{HD}$ we refer to Sheather and Marron [19]. Let us generate $n = 100$ independent random variables $X_1, \ldots, X_n$ from $F(x) = x^3, 0 \leqslant x \leqslant 1$. Taking $\alpha = 100$, we estimate (see, Figure 2 (b)) the corresponding quantile function $Q(x) = x^{1/3}, 0 \leqslant x \leqslant 1$, (the dashed line) by means of $\hat{Q}_\alpha$ (the solid line) and by $\hat{Q}_{HD}$ (the dashed-dotted line), defined in (4.6) and (4.7), accordingly. Through simulations we conclude that the asymptotic behavior of the moment-recovered estimator $\hat{Q}_\alpha$ and the Harrell-Davis estimator $\hat{Q}_{HD}$ are similar. The MSE and other properties of $\hat{Q}_\alpha, \hat{q}_\alpha$, and $\hat{q}^*_\alpha$ will be presented in a separate article.

Example 4.1 (continued). Assume now that we want to recover pdf of the distribution G studied in the Example 4.1 via the moments $\nu_{j,G} = 1/(j+1)^2, j \in \mathbb{N}$. On the Figure 3 (a) we plotted the curves of the moment-recovered density $f_{\alpha,\nu}$ (the solid line) defined by (3.11) and $g(x) = G'(x) = -\ln x,\ 0 \leqslant x \leqslant 1$ (the dashed line), respectively. Here we took $\alpha = 50$ and $M = 200$.

Example 4.2 (continued). Now let us recover the pdf $f(x) = -9x^2 \ln x, 0 \leqslant x \leqslant 1$, of distribution F defined in Example 4.2 where $\nu_{j,F} = 9/(j+3)^2, j \in \mathbb{N}$. We applied the both approximants $f_{\alpha,\nu}$ and $f^*{}_{\alpha,\nu}$ defined in (3.11) and (3.14), respectively.



(a) (b)

FIG 3. *(a) Approximation of $g(x) = -\ln x$ by $f_{\alpha,\nu}$ and (b) Approximation of $f(x) = -9x^2 \ln x$ by $f_{\alpha,\nu}$ and $f^*_{\alpha,\nu}$*

Namely, we calculated the values of $f_{\alpha,\nu}$ and ${f^*}_{\alpha,\nu}$ at the points $x = k/\alpha$, $k = 1, 2, \ldots, \alpha$. On the Figure 3 (b) we plotted the curves of $f_{\alpha,\nu}$ (the blue dashed-dotted line), and $f^*_{\alpha,\nu}$ (the red solid line), and f (the black dashed line). Here, we took $\alpha = 50$ and $M = 200$ when calculating $f_{\alpha,\nu}$ and $\alpha = 32$ in $f^*_{\alpha,\nu}$. One can see that the performance of $f^*_{\alpha,\nu}$ with $\alpha = 32$ is better than the performance of $f_{\alpha,\nu}$ with $\alpha = 50$ and $M = 200$.

After conducting many calculations of moment-recovered approximants for several models we conclude that the accuracy of the formulas (2.2) and (3.11) are not as good as the ones defined in (3.13) and (3.14) in the Hausdorff case. On the other hand, the constructions (2.2) and (3.11) could be useful in the Sieltjes moment problem as well.

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