IMS Collections
Vol. 0 (2009) 347–351
© Institute of Mathematical Statistics, 2009
arXiv: math.PR/00000019

A Note on the Investment Proportions of a Minimum-Variance Equity Portfolio.

Wilhelmine von Türk[1,*]

American Century Investments

Abstract: It is shown that the capitalization-weighted portfolio is mathematically required to coincide with the minimum-variance portfolio, provided both portfolios are defined with respect to the same (arbitrary) collection of equities having linearly independent returns. This result is a logical consequence of the law of iterated expectations and has important implications for equity return covariance structure.

Contents

1. Introduction

Samuelson (1965) was the first to recognize the relevance of the Law of Iterated Expectations (LIE) in the area of finance. His celebrated paper shows how this law can be applied in the context of the price of a single risky asset. Essentially, the LIE reveals that forecasting error is not predictable. An illuminating discussion of this law can be found, e.g., in Campbell, Lo and MacKinlay (1997).

This paper extends Samuelson's insight by developing an application of the LIE in the context of an arbitrary collection of equity returns. Specifically, it will be shown that the LIE implies that the capitalization-weighted portfolio is mathematically required to coincide with the minimum-variance portfolio, provided both portfolios are defined with respect to the same (arbitrary) collection of equities having linearly independent returns. This result has important implications for equity return covariance structure, as summarized in technical appendix A.

[1]43 Del Mar Avenue, Berkeley, CA 94708, email: wvonturk@sbcglobal.net
*The author gratefully acknowledges her intellectual indebtedness to the tradition of the Department of Statistics, University of California at Berkeley, where she completed the dissertation with the guidance and advice of Professor Erich L. Lehmann. The responsibility for any shortcomings in the present paper rests squarely on the shoulders of the author.
AMS 2000 subject classifications: Primary 91B28; secondary 62H25
Keywords and phrases: minimum-variance portfolio, capitalization-weighted portfolio, equity return covariance structure, Law of Iterated Expectations, equity portfolio optimization.

The paper is organized as follows. Section 2 introduces the notation and presents a sufficient condition for proving the main result. The population view of the surprises (forecasting errors) is developed in Section 3. This population view yields an important building block for the proof of the main result. Section 4 applies the LIE in the context of a cross-section of equities, and exhibits the implication of this law from which the main result logically flows. The proof of the main result is summarized in Section 5. Section 6 concludes.

Technical appendix A summarizes the implications of this paper's main result for equity return covariance structure.

2. Notation and Preliminary Results

Consider an arbitrary collection of n equities with linearly independent returns. Denote their returns by $\mathbf{R} = (R_1 \dots R_n)'$ and denote the covariance matrix of $\mathbf{R}$ by Σ. Since the returns are assumed to be linearly independent, Σ is nonsingular, so that its inverse Σ^{-1} is well defined. Let $\mathbf{1}$ denote the n x 1 vector all of whose elements are equal to 1.

Let $\mathbf{M} = (M_1 \dots M_n)'$ denote the market capitalizations of the equities. Let $w_i = M_i/\mathbf{M}'\mathbf{1}$ and let $\mathbf{w} = (w_1 \dots w_n)'$. The vector $\mathbf{w}$ corresponds to the *capitalization weights*. Note that all of the elements of $\mathbf{w}$ are positive. Let $\mathbf{p} = (p_1 \dots p_n)'$ denote an n x 1 vector of constants such that $\mathbf{p}'\mathbf{1} = 1$. Then the elements of $\mathbf{p}$ correspond to the investment proportions of a portfolio fully invested in the n equities, and $\mathbf{p}'\mathbf{R}$ denotes its return. If $\mathbf{p} = \mathbf{w}$, the investment proportions are those of the capitalization-weighted portfolio. The expression $\mathbf{w}'\mathbf{R}$ denotes the return of the capitalization-weighted portfolio.

Among all possible fully invested portfolios that can be formed from this collection of equities, there is one portfolio whose return has minimum variance. This portfolio is the minimum-variance portfolio. It is well known that, if Σ is nonsingular, the investment proportions of the minimum-variance portfolio are given by $\Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$ (see, e.g., Roll (1977), Campbell, Lo and MacKinlay (1997) or Grinold and Kahn (2000)).

The desired result is that

$$\mathbf{w} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}.$$

For the purpose of finding a proof of this result, it is convenient to define

$$\beta_i = \mathrm{Cov}(R_i, \mathbf{w}'\mathbf{R})/\mathrm{Var}(\mathbf{w}'\mathbf{R}).$$

Let $\boldsymbol{\beta} = (\beta_1 \dots \beta_n)'$. Then

$$\boldsymbol{\beta} = \Sigma\mathbf{w}/\mathbf{w}'\Sigma\mathbf{w}.$$

Note that $\mathbf{w}'\mathbf{1} = 1$, so that

$$\mathbf{w}'\Sigma\mathbf{w} = 1/\boldsymbol{\beta}'\Sigma^{-1}\mathbf{1}$$

and

$$\mathbf{w} = \Sigma^{-1}\boldsymbol{\beta}/\boldsymbol{\beta}'\Sigma^{-1}\mathbf{1}.$$

With this notation it is possible to establish the following preliminary result.

Lemma 2.1. *The following assertions are equivalent*:

(i) $\mathbf{w} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$

(ii) $\boldsymbol{\beta} = \mathbf{1}$

Proof. $\mathbf{w} = \Sigma^{-1}\boldsymbol{\beta}/\boldsymbol{\beta}'\Sigma^{-1}\mathbf{1}$ implies that $\Sigma\mathbf{w} = \boldsymbol{\beta}/\boldsymbol{\beta}'\Sigma^{-1}\mathbf{1}$, and $\mathbf{w} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$ implies that $\Sigma\mathbf{w} = \mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$. Therefore, $\mathbf{w} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$ if and only if $\boldsymbol{\beta} = c\mathbf{1}$ for $c = \boldsymbol{\beta}'\Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$. Since $\mathbf{w}'\boldsymbol{\beta} = \mathbf{w}'\mathbf{1} = 1$, this constant c is equal to 1.

Next, consider an investment period which begins at time $t = 0$ and ends at time $t = T$. Note that at time $t = 0$, $\mathbf{w}$ is a vector of known constants while $\mathbf{R}$ is a vector of random variables. Let $\mathrm{E}[\mathbf{R}]$ denote the n x 1 vector whose elements correspond to $\mathrm{E}[R_i]$, where $\mathrm{E}[R_i]$ is the expected value of R_i at time $t = 0$ ($i = 1, \ldots, n$). The expression $\mathrm{E}[R_i]$ corresponds to the best forecast at time $t = 0$ of the realized value of R_i observed at time $t = T$.

Now shift attention from the returns R_i to the surprises S_i, where $S_i = R_i - E[R_i]$. The surprises correspond to *forecasting errors*. Let $\mathbf{S} = (S_1, \ldots, S_n)'$, let $\mathcal{A}$ denote the σ-field induced by $\mathbf{S}$, and let $\mathcal{F}$ denote the σ-field induced by $\mathbf{w}'\mathbf{S}$. Since $\mathbf{w}'\mathbf{S}$ is a function of $\mathbf{S}$, $\mathcal{F} \subset \mathcal{A}$. The conditional expectation of S_i given $\mathbf{w}'\mathbf{S}$ can then be written as $\mathrm{E}^{\mathcal{F}}[S_i]$, which is defined almost surely (a.s.) in the sense that any two versions agree, except possibly on a null set in $\mathcal{A}$. With this notation, it is possible to simplify the task of proving the desired result as follows.

Lemma 2.2. *For the purpose of showing that* $\mathbf{w} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$, *it is sufficient to show that* $\mathrm{E}^{\mathcal{F}}[S_i] = \mathbf{w}'\mathbf{S}$ *(a.s.)* $(i = 1, \ldots, n)$.

Proof. In light of Lemma 2.1, it is sufficient to show that $\mathrm{E}^{\mathcal{F}}[S_i] = \mathbf{w}'\mathbf{S}$ (*a.s.*) $(i = 1, \ldots, n)$ implies that $\boldsymbol{\beta} = \mathbf{1}$.

By definition,

$$\mathrm{Cov}(R_i, \mathbf{w}'\mathbf{R}) = \mathrm{E}[S_i\mathbf{w}'\mathbf{S}]$$

and

$$\mathrm{Var}(\mathbf{w}'\mathbf{R}) = \mathrm{E}\left[(\mathbf{w}'\mathbf{S})^2\right].$$

Therefore it suffices to show that $\mathrm{E}^{\mathcal{F}}[S_i] = \mathbf{w}'\mathbf{S}$ (*a.s.*) $(i = 1, \ldots, n)$ implies that

$$\mathrm{E}[S_i\mathbf{w}'\mathbf{S}] = \mathrm{E}\left[(\mathbf{w}'\mathbf{S})^2\right] \; (i = 1, \ldots, n).$$

This is easily accomplished by recalling the usual properties of conditional expectation operators, which yields

$$\mathrm{E}[S_i\mathbf{w}'\mathbf{S}] = \mathrm{E}\left[\mathbf{w}'\mathbf{S}\mathrm{E}^{\mathcal{F}}[S_i]\right],$$

so that $\mathrm{E}^{\mathcal{F}}[S_i] = \mathbf{w}'\mathbf{S}$ (a.s.) implies $\mathrm{E}[S_i\mathbf{w}'\mathbf{S}] = \mathrm{E}\left[(\mathbf{w}'\mathbf{S})^2\right]$ $(i = 1, \ldots, n)$.

Before it can be shown that the assertion $\mathrm{E}^{\mathcal{F}}[S_i] = \mathbf{w}'\mathbf{S}$ (a.s.) $(i = 1, \ldots, n)$ flows logically from the LIE, it is first necessary to develop the population view of the surprises (forecasting errors). This is done in the next section.

3. The Population View

Consider an arbitrary collection of n publicly traded companies and an investment period which begins at time $t = 0$ and ends at time $t = T$. The development of the population view of the surprises depends on the operation of repricing the shares of each company at the beginning of the investment period, while adjusting the shares outstanding so as to leave the market capitalization of each company unchanged. This operation, very familiar to equity investors, guarantees that there is no loss of generality in repricing the shares of each company at the beginning of the investment period to have a value of one dollar.

As in Section 2, let M_i denote the market capitalizations at the beginning of the investment period ($i = 1, \dots, n$). If the price of one share is one dollar (and if the market capitalizations are rounded to the nearest dollar), then M_i corresponds to the number N_i of shares outstanding for the i-th company. Each share of the i-th company has return R_i and associated surprise (forecasting error) S_i. Therefore, at the end of the investment period, the observed values s_i of the random variables S_i form a population in which s_i occurs with frequency N_i. The capitalization weights w_i then correspond to the relative frequencies with which the s_i are observed. This population has mean $\mathbf{w}'\mathbf{s}$, where $\mathbf{s} = (s_1, \dots, s_n)'$ is the observed value of the random vector $\mathbf{S} = (S_1, \dots, S_n)'$.

It is well known that the population mean is the expected value of a randomly selected element from that population. From the standpoint of time $t = T$, the population mean is observed with certainty, while from the standpoint of time $t = 0$, the population mean corresponds to the random variable $\mathbf{w}'\mathbf{S}$. This means that

$$\mathrm{E}_T[\text{a randomly selected surprise}] = \mathbf{w}'\mathbf{s},$$

where "E_T" is to be read as "the expected value at time $t = T$, conditional on the observed value $\mathbf{w}'\mathbf{s}$ of $\mathbf{w}'\mathbf{S}$." Translating this into σ-field notation yields the following lemma.

Lemma 3.1. $\mathrm{E}^{\mathcal{F}}$[a randomly selected surprise]$= \mathbf{w}'\mathbf{S}$ $(a.s.)$, *where* $\mathcal{F}$ *denotes the* σ*-field induced by* $\mathbf{w}'\mathbf{S}$.

As will be seen in Section 5, Lemma 3.1 is an important building block for the proof of the main result.

4. The Law of Iterated Expectations

A key insight provided by the LIE is that forecasting error is not predictable. The present paper contemplates an arbitrary collection of n equities with returns R_i and associated forecasting errors $S_i = R_i - \mathrm{E}[R_i]$ ($i = 1, \dots, n$). In this context, the LIE implies that the expected value of any surprise S_i corresponds to the expected value of a *randomly selected surprise*. From the standpoint of time $t = 0$, this yields the set of equations

$$\mathrm{E}[S_i] = \mathrm{E}[\text{a randomly selected surprise}] \ (i = 1, \dots, n).$$

Conditional on $\mathbf{w}'\mathbf{S}$, the LIE similarly implies the following lemma.

Lemma 4.1. $\mathrm{E}^{\mathcal{F}}[S_i]$=$\mathrm{E}^{\mathcal{F}}$[a randomly selected surprise] $(a.s.)$ $(i = 1, \dots, n)$, *where* $\mathcal{F}$ *denotes the* σ*-field induced by* $\mathbf{w}'\mathbf{S}$.

Lemma 4.1 exhibits the implication of the LIE from which the main result of the present paper logically flows, as summarized in the next section.

5. Proof of the Main Result

The main result is stated as the proposition below.

PROPOSITION: *For an arbitrary collection of n equities having linearly independent returns,* $\mathbf{w} = \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1}$.

Proof. From Lemma 4.1,

$$\mathrm{E}^{\mathcal{F}}[S_i] = \mathrm{E}^{\mathcal{F}}[\text{a randomly selected surprise}] \text{ (a.s.) } (i = 1, \ldots, n).$$

From Lemma 3.1,

$$\mathrm{E}^{\mathcal{F}}[\text{a randomly selected surprise}] = \mathbf{w}'\mathbf{S} \text{ (a.s.)}.$$

Combining Lemmas 4.1 and 3.1 yields

$$\mathrm{E}^{\mathcal{F}}[S_i] = \mathbf{w}'\mathbf{S} \text{ (a.s.) } (i = 1, \ldots, n).$$

In light of Lemma 2.2, this completes the proof.

6. Conclusion

It has been shown that the capitalization-weighted portfolio is mathematically required to coincide with the minimum-variance portfolio, provided both portfolios are defined with respect to the same (arbitrary) collection of equities having linearly independent returns. This result is a logical consequence of the LIE, and has important implications for equity return covariance structure, as summarized in technical appendix A.

Technical Appendix A

The main result of this paper has important implications for equity return covariance structure, as summarized in the following proposition.

PROPOSITION: *For any collection of n equities with linearly independent returns, the covariance matrix Σ of $\boldsymbol{R} = (R_1, \ldots, R_n)'$ is of the form*

$$\Sigma = \mathbf{1}\mathbf{1}'k + \mathbf{U}^2$$

where $\mathbf{U^2}$ is a diagonal matrix such that the i-th diagonal element is positive and inversely proportional to M_i ($i = 1, \ldots, n$), and where k corresponds to a constant which can be positive, negative or zero.

Note that for positive values of k, general equity return covariance structure exhibited in the proposition above corresponds to the covariance matrix of a 1-factor model in which

(i) the factor loadings of the unique common factor are all equal to one; and
(ii) the variances of the specific factors are inversely proportional to the market capitalizations of the equities.

References

[1] Campbell, J. Y., Lo, A. W. and MacKinlay, A. C. (1997). The Econometrics of Financial Markets. Princeton University Press, Princeton, New Jersey.

[2] Fernholz, E. R. (2002). Stochastic Portfolio Theory. Springer Science + Business Media, Inc.

[3] Grinold, R. C. and Kahn, R. N. (2000). Active Portfolio Management. McGraw-Hill, New York.

[4] Roll, R. (1977). A critique of the asset pricing theory's tests. *Journal of Financial Economics* **4**, 129–176.

[5] Samuelson, P. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review* **6**, 41–49.