An Introduction to Econometrics

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Chapter 1. What is econometrics?

It is the application of statistical theories to economic ones for the purpose of forecasting future trends.
It takes economic models and tests them through statistical trials. The results are then compared and contrasted against real life examples.

Chapter 2. Demand and Supply

Demand: A consumer's desire and willingness to pay for a good or service.
Supply: The total amount of a good or service available for purchase by consumers.
They are all affected by the market price.

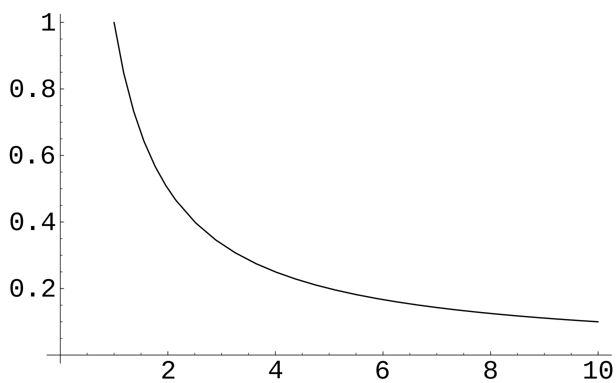
Demand Function: $q = f(p)$, here p denotes for the price of a commodity and q represents the demand of consumers.
Supply Function: $q = g(p)$, here p denotes for the price of a commodity and q represents the supply of producers.

Postulates about the market:

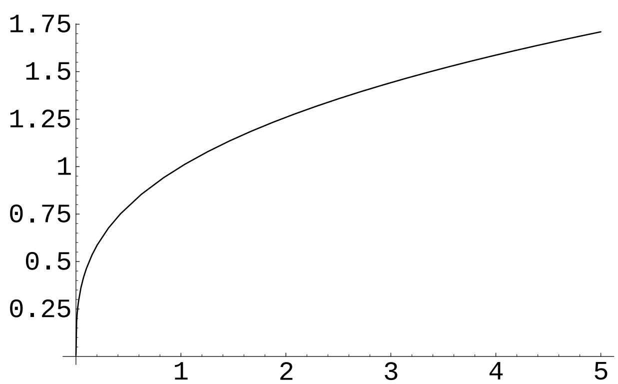
Law of Downward Sloping Demand: When the price goes up, the demand diminishes.
Law of Upward Sloping Supply: The higher is the price, the more is the supply.

Law of Demand and Supply:
When demand is higher than supply, the price goes up; otherwise, the price goes down.

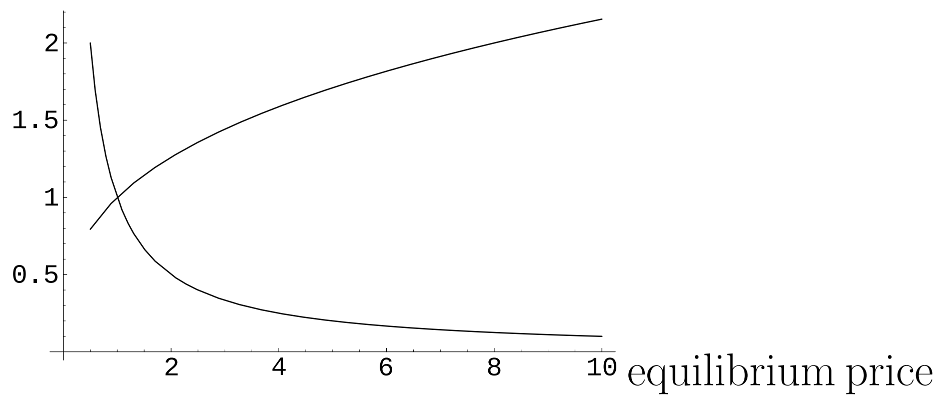
Geometric Expression of Demand and Supply Function:



$q = f(p)$



$q = g(p)$



Chapter 3. Utility

Utility: The satisfaction obtained by a consumer from consuming a good or service.
Marginal Utility: The additional satisfaction obtained by a consumer from consuming one more unit of a good or service.
Marginal analysis is a method used in economics similar to the differential method in mathematics. If we denote $y = f(x)$, x is an integer, then $f(n) - f(n-1)$ is called the marginal value of y at $x = n$. If x can be continuous value, and f is differentiable, then dy/dx is the marginal value of y at x.

Postulate of Marginal Utility:
Law of Diminishing Marginal Utility: When the consuming quantity x increases, the marginal util-

ity dy/dx decreases.

Chapter 4. Production Function

Production Function: Suppose that $x_1, ..., x_n$ are input levels of n production factors, production function is the biggest output of this kind of input combination $(x_1, ..., x_n)$.
If $f(kx_1, ..., kx_n) > kf(x_1, ..., x_n)$, then this production is called increasing-on-production scale.
If $f(kx_1, ..., kx_n) = kf(x_1, ..., x_n)$, then this production is called invariable-on-production scale.
If $f(kx_1, ..., kx_n) < kf(x_1, ..., x_n)$, then this production is called decreasing-on-production scale.

Chapter 5. Kuhn-Tucker Condition

Suppose $f(x_1, ..., x_n), g_i(x_1, ..., x_n), h_j(x_1, ..., x_n), i = 1, ..., l, j = 1, ..., m$ are $1 + l + m$ continuous differentiable functions in $X \subseteq \Re^n$.
Let us consider the maximization problem:

$$max f(x_1, ..., x_n)$$
$$\text{s.t. } g_i(x_1, ..., x_n) = 0, i = 1, ..., l$$
$$h_j(x_1, ..., x_n) \leq 0, j = 1, ..., m$$

If $(x_1^*, ..., x_n^*) \in X$ is the optimum solution, and satisfies the regularity, that is, at the point $x^* = (x_1^*, ..., x_n^*)$, all the ∇g_i and ∇h_j such that $h_j(x^*) = 0$ are linear independent, then there exist l real numbers $\lambda_1, ..., \lambda_l$ and m nonnegative real numbers $\mu_1, ..., \mu_m$, such that

$$\nabla[f(x) - \Sigma_{i=1}^{l} \lambda_i g_i(x) - \Sigma_{j=1}^{m} \mu_j h_j(x)]|_{x=(x_1^*,...,x_n^*)} = \overrightarrow{0} \quad (1)$$

$$\Sigma_{j=1}^{m} \mu_j h_j(x_1^*, ..., x_n^*) = 0 \quad (2)$$

Here, ∇ is the gradient operator:

$$\nabla\varphi(x) = (\frac{\partial\varphi}{\partial x_1}, ..., \frac{\partial\varphi}{\partial x_n})^T$$

and $\overrightarrow{0} = \underbrace{(0, ..., 0)}_{n}^T$

(1) and (2) are called Kuhn-Tucker Condition. We have similar conclusion about the minimization problem:

$$min f(x_1, ..., x_n)$$
$$\text{s.t. } g_i(x_1, ..., x_n) = 0, i = 1, ..., l$$
$$h_j(x_1, ..., x_n) \geq 0, j = 1, ..., m$$

Chapter 6. Utility Function

Suppose we have n commodities in the market, x_i is the consuming quantity of the ith commodity of

the consumer, $i = 1, 2, ..., n$.
we call the vector

$$x = \overrightarrow{(x_1, x_2, ..., x_n)}$$

consuming vector(or consuming planning) of the consumer.

$$X = \{x | x \geq 0\}$$

is called the consuming set.
If for all the consuming planning in X, there is a semi-order $\succeq$ which satisfies the following four postulates A1,A2,A3,A4, then this consumer is called rational.

A1(Complete)
$\forall x, y \in X$,either $x \succeq y$ or $y \succeq x$
A2(Reflective)
$\forall x \in X, x \succeq x$
A3(Transitive)
$\forall x, y, z \in X$,if $x \succeq y, y \succeq z$,then $x \succeq z$
A4(Continuous)
$\forall y \in X, x^k \in X$, if $x^k \succeq y$, and $x^k \to \overline{x}(k \to \infty)$, then $\overline{x} \in X$, and $\overline{x} \succeq y$

For any $x, y \in X$, the semi-order $x \succeq y$ means that the consumer deems that the plan x is not

worse than y.
If there exists a function

$$u : X \to R$$

such that for all $x, y \in X$,

$$x \succeq y \Leftrightarrow u(x) \geq u(y)$$

then $u(x)$ is called a utility function of this consumer(relative to this semi-order $\succeq$).
Obviously we have several properties of the utility function:
Property 1. $x \sim y \Leftrightarrow u(x) = u(y)$ ($x \sim y$ means that $x \succeq y$ and $y \succeq x$)
Property 2. $x \succ y \Leftrightarrow u(x) > u(y)$, here $x \succ y$ means that the consumer thinks that "x is better than y", that is $x \succeq y$, but $x \sim y$ does not hold.

The utility function exists under certain conditions.
Debreu Theorem:
If the consumer's semi-order $\succeq$ satisfies A1-A4, then there exists a continuous utility function.
(Refer to <<International Economic Review 5>> Page285-293)
Theorem(Non-Uniqueness):
Suppose $u(x)$ is a utility function of $\succeq$, and f :

$R \to R$ is any increasing function, then $f(u(x))$ is also a utility function of $\succeq$.

For further discussion, we put forward some postulates about the semi-order $\succeq$:
A5(Local Unsaturated)$\forall x \in X, \epsilon > 0, \exists y \in X$ such that $\|y - x\| < \epsilon, y \succ x$
A6 (Convex) $\forall x, y, z \in X, x \succeq z, y \succeq z$, then $\forall \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \succeq z$
A7(Strict Convex) $\forall x, y, z \in X, x \neq y, x \succeq z, y \succeq z$,then $\forall \lambda \in (0, 1)$, we have $\lambda x + (1 - \lambda)y \succ z$

Now we consider the maximization problem (P_1) of the utility function:

$$\max u(x)$$
$$\text{s.t } px \leq m$$
$$x \in X = \{x | x \geq 0\}$$

Here, $x = (x_1, ..., x_n)^T$ is the consuming vector of this consumer, and $u(x) = u(x_1, ..., x_n)$ is the utility function of this consumer. $p = (p_1, ..., p_n)$ is the price vector. p_i is the price of the ith commodity,$i = 1, 2, ..., n$. m is the available money of this consumer.
The maximization problem tries to find that how many should this consumer buy in order to get the

maximum utility.

When A5 holds, the problem's optimum solution x^* satisfies $px^* = m$.
This is because, according to Bolzano-Weierstras theorem, x^* does exit. If $px^* < m$, since $x^* \in X$, using the A5, we can find $\epsilon > 0$ and $y \in X$ such that

$$\|y - x^*\| < \epsilon,\ py \leq m \text{ and } y \succ x^*$$

so $u(y) > u(x^*)$, a contradiction with the property of x^*.
So the maximization problem can be rewrote as the following maximization problem (P_1'):

$$\begin{gathered} \max u(x) \\ \text{s.t } px = m \\ x \in X \end{gathered}$$

Theorem: Suppose that $\succeq$ satisfies A7, then its utility function is strictly quasiconcave. That is to say, $\forall x, y \in X, x \neq y, \lambda \in (0,1)$, we have $u(\lambda x + (1-\lambda)y) > min(u(x), u(y))$.
Proof: For any $x, y \in X, x \neq y$, assuming $x \succeq y$, then $u(x) \geq u(y)$.
Now for any $\lambda \in (0,1)$, according to A7, we have $\lambda x + (1-\lambda)y \succ y$. So $u(\lambda x + (1-\lambda)y) > u(y) =$

$min(u(x), u(y))$.

Theorem: Suppose that $\succeq$ satisfies A7, then the optimum solution of (P_1') is unique.
Proof: Suppose $x^* \neq x^{**}$ are both maximum solution, since the set $B = \{x | x \in X, px = m\}$ is convex, so for any $\lambda \in (0, 1)$, the point $\lambda x^* + (1-\lambda)x^{**} \in B$ and using the previous theorem, $u(\lambda x^* + (1-\lambda)x^{**}) > min(u(x^*), u(x^{**})) = u(x^*) = u(x^{**})$, which is a contradiction with that x^* and x^{**} are both maximization points.

In the model of maximization of utility, optimum solution x^* is a vector function of p and m, denote as

$$x^* = x(p, m)$$

then the maximum $u(x^*)$ is also a function of p and m, denote as

$$v(p, m) = u(x^*) = u(x(p, m))$$

we call $v(p, m)$ indirect utility function of this consumer.
$v(p, m)$ has following important properties:
1.If $p_j^1 \geq p_j^2$, then $v(p_1^1, ..., p_n^1, m) \leq v(p_1^2, ..., p_n^2, m)$
2.If $m_1 \geq m_2$, then $v(p, m_1) \geq v(p, m_2)$

$3.v(p, m) = v(tp, tm), \forall t > 0$
$4.v(p, m)$ is continuous when $p > 0, m > 0$

Now we return to the problem (P_1'), which is a nonlinear layout. Using the Kuhn-Tucker condition, we know that there exists a constant λ at the optimum solution(maximum point)x^*(suppose it satisfies the regularity), such that

$$\nabla[u(x) - \lambda px]|_{x=x^*} = \overrightarrow{0}$$

that is

$$\frac{\partial u(x^*)}{\partial x_i} - \lambda p_i = 0, i = 1, ..., n$$

or

$$\frac{1}{p_i}\frac{\partial u(x^*)}{\partial x_i} = \lambda, i = 1, ..., n$$

Since $\frac{1}{p_i}$ denotes the quantity of ith commodity which the consumer can buy using unit money, and $\frac{\partial u(x^*)}{\partial x_i}$ is the marginal utility of the ith commodity, the left hand side of the above equality is marginal utility of unit incoming. The equality shows that, at the maximum point (x^*), all the n commodities' marginal utilities of unit incoming equal to λ.

Chapter 7. Demanding Function

(P_1)'s optimum solution's expression as parameters (p, m) is $x^* = x(p, m)$
It is demanding function, called Marshall Demanding Function.
It has following properties:
1.Roy Equality:

$$x_j(p, m) = -\frac{\frac{\partial v(p,m)}{\partial p_j}}{\frac{\partial v(p,m)}{\partial m}}, j = 1, ..., n$$

2.Zero Degree Homogeneity, that is $x(tp, tm) = x(p, m), \forall t > 0$
3.Symmetry, that is

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial m} = \frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial m}, i, j = 1, ..., n$$

4.Inequality $\frac{\partial x_i}{\partial p_i} + x_i \frac{\partial x_i}{\partial m} \leq 0, i = 1, ..., n$

The task of (P_1') is to find the maximum utility in condition of fixed incoming m. Its dual problem is to find the minimum expenditure in condition of fixed utility u. Thus let us consider the following nonlinear layout (P_2'):

$$\min px$$
$$\text{s.t } u(x) = u$$
$$x \in X$$

Applying Kuhn-Tucker condition again, there exists a real number λ at the optimum solution $\hat{x}$, such that

$$\nabla[px - \lambda u(x)]|_{x=\hat{x}} = \overrightarrow{0}$$

that is

$$p_i - \lambda \frac{\partial u(\hat{x})}{\partial x_i} = 0, i = 1, ..., n$$

Rewrite the optimum solution $\hat{x}$ as a vector function of parameters p and u:

$$\hat{x} = h(p, u)$$

or

$$\hat{x}_i = h_i(p, u), i = 1, ..., n$$

It is called Hicks Demanding Function.

We call the optimum solution of (P_2')(that is the minimum of px)

$$e(p, u) = p\hat{x} = \Sigma_{i=1}^{n} p_i h_i(p, u)$$

payout function. It is a scalar function.
It has following properties:
1.$e(p, u)$ is a nondecreasing function of p.
2.$e(p, u)$ is a first degree homogeneous function of p. That is to say $e(tp, u) = te(p, u)$.
3.$e(p, u)$ is a concave function of p.
4.$e(p, u)$ is a continuous function of p.

Now let us discuss the two dual nonlinear layout:
(P_1)

$$\max u(x)$$
$$\text{s.t.} px \leq m$$

(P_2)

$$\min px$$
$$\text{s.t.} u(x) \geq u$$

Suppose the semi-order satisfies A4 and A5, and both of the problems have optimum solutions. We have:
Theorem: Suppose x^* is (P_1)'s optimum solution, then x^* is also (P_2)'s optimum solution, where $u = u(x^*)$.
Proof. If not, then suppose x' is (P_2)'s optimum solution when $u = u(x^*)$, then

$$px' < px^*$$
$$u(x') \geq u(x^*)$$

From A5, we know that there exists a x'' close enough with x', such that

$$px'' < px^* = m$$

and

$$u(x'') > u(x^*)$$

a contradiction, since x^* is (P_1)'s optimum solution.

Theorem Suppose x^* is (P_2)'s optimum solution, then x^* is also (P_1)'s optimum solution, where $m = px^*$ and assuming $m > 0$.
Proof: If not, suppose x' is (P_1)'s optimum solution when $m = px^*$, then

$$u(x') > u(x^*)$$
$$px' = px^*$$

Since the semi-order satisfies A4, then there exists $t \in (0, 1)$, such that (tx') satisfies

$$p(tx') < px^*$$
$$u(tx') > u(x^*)$$

a contradiction, since x^* is the optimum solution of (P_2).

Summarize the above results, we have the following four equalities:

$$e(p, v(p, m)) = m$$
$$v(p, e(p, u)) = u$$
$$x(p, m) = h(p, v(p, m))$$
$$h(p, u) = x(p, e(p, u))$$

Chapter 8.Cost Function

Suppose there are n production factors in some production process, the production function is $f(x_1, ..., x_n)$, here x_i denotes the input level of ith production factor, $i = 1, ..., n$, and the price of the ith production factor is $p_i, i = 1, ..., n$, then the cost function of producers is

$$C = p_1 x_1 + ... + p_n x_n + b = px + b$$

here, b is the fixed cost of this production process, a positive constant.

Let's consider the minimization problem (P_3) of producer's cost:

$$\min\ C(x) = px + b$$
$$\text{s.t } f(x) = q$$

here, q is the given output level. We want to find the minimum production factor combination in the condition of given output level.

According to Kuhn-Tucker Condition, there exits a real constant λ at the optimum solution x^* (suppose it satisfies regularity), such that

$$\nabla[px + b - \lambda(f(x) - q)]|_{x=x^*} = \overrightarrow{0}$$

That is

$$p_i - \lambda \frac{\partial f(x^*)}{\partial x_i} = 0, i = 1, ..., n$$

Denote

$$\frac{\partial f}{\partial x_i} = f_i, i = 1, ..., n$$

then

$$p_i - \lambda f_i(x^*) = 0, i = 1, ..., n$$

or

$$\frac{f_1(x^*)}{p_1} = ... = \frac{f_n(x^*)}{p_n} = \lambda^{-1}$$

The optimum solution $x = x(p, q)$ is called demand function of production factors.
Plug the demand function of production factors into (P_3), we have

$$C = px(p, q) + b,$$

which is the cost function of variable p and q.
The cost function $C(p_1, ..., p_n, q)$ has following properties:
1. It is monotone about the factor price. That is to say, if $p_i^1 \geq p_i^2$, for some i, then $C(p_1, ..., p_i^1, ..., p_n, q) \geq C(p_1, ..., p_i^2, ..., p_n, q)$
2. It is concave about the factor price. That is to say, $C(\lambda p_1^1 + (1-\lambda)p_1^2, .., \lambda p_n^1 + (1-\lambda)p_n^2, q) \geq \lambda C(p_1^1, ..., p_n^1, q) + (1-\lambda)C(p_1^2, ..., p_n^2, q)$ for every

$\lambda \in [0, 1]$
3. It is monotone about the output level. That is to say, if $q_1 \geq q_2$, then $C(p_1, ..., p_n, q_1) \geq C(p_1, ..., p_n, q_2)$

Chapter 9. Supply Function

Suppose the production function of a production process is $f(x_1, ..., x_n)$, here x_i is the input level of ith production factor,$i = 1, ..., n$, and suppose the price of the ith production factor is $p_i, i = 1, ..., n$, then the income of the producer is

$$R = p_0 f(x_1, ..., x_n) = p_0 f(x)$$

where, p_0 is the price of the production, $x = (x_1, ..., x_n)^T$ is the input level of the production factors.
The cost of the producer is

$$C = p_1 x_1 + ... + p_n x_n + b = px + b$$

here b is the fixed cost of the production process, a positive constant.
So the profit of this producer is

$$\pi = R - C = p_0 f(x) - px - b$$

Let us consider the maximization problem of the producer's profit:

$$\max\ \pi(x) = p_0 f(x) - px - b$$
$$\text{s.t } x \in X$$

Suppose the optimum solution is x^* (assuming that it satisfies the regularity), then x^* satisfies

$$p_0(\frac{\partial f}{\partial x_i})_{x=x^*} - p_i = 0, i = 1, ..., n$$

or

$$f_i(x^*) = \frac{p_i}{p_0}, i = 1, ..., n$$

Chapter 10. Equilibrium

Equilibrium: The state where market supply and demand balance each other and, therefore, prices are stable.
Now let us take a look at the simplest equilibrium in econometrics–Walras Equilibrium.
Suppose there are n different commodities in the market, and m different consumers. In the beginning of the trade, the ith consumer's hold vector of commodities is

$$w_i = (w_1^i, ..., w_n^i)^T,$$

here, the w_j^i is the quantity of jth commodity held by the ith consumer,$j = 1, ..., n, i = 1, ..., m$.
Denote the price of the jth commodity as $p_j, j = 1, ..., n$, then these m consumers trade commodities between each other according to the price above.

In the end of the trade, the ith consumer has commodities

$$x^i = (x_1^i, ..., x_n^i)^T, i = 1, ..., m$$

We call this $n \times m$ matrix

$$x = (x^1, ..., x^m)$$

a distribution. If the condition

$$\Sigma_{i=1}^m x^i = \Sigma_{i=1}^m w_i$$

holds, then x is called an attainable distribution, which means that the commodities do not vanish or increase during the trade.

In the market above, all the consumers do not work, they just trade in order to make their utilities maximum.

Denote the ith consumer's utility function as

$$u_i(x^i) = u_i(x_1^i, ..., x_n^i), i = 1, ..., m$$

Naturally, we have the following m problems $(P_i), i = 1, ..., m$

$$\begin{array}{c} \max\ u_i(x^i) \\ \text{s.t. } px^i = pw^i \\ x^i \in X_i, \end{array}$$

here, X_i is the consuming planning set of the ith consumer. Normally, it is

$$X_i = \{x^i | x_j^i \geq 0, j = 1, ..., n\}, i = 1, ..., m$$

Suppose each of the maximization problem above has unique optimum solution, and denote them as

$$x^{i^*} = x^i(p, pw^i)$$

it is called Marshall Demand Function of the ith consumer,$i = 1, ..., m$
Denote $z(p) = \Sigma_{i=1}^m [x^i(p, pw^i) - w^i]$
Its component is $z_j(p) = \Sigma_{i=1}^m [x_j^i(p, pw^i) - w_j^i]$
Obviously, it represents the total excess of demand in the market. Every component represents the excess demand of this commodity.

For given price $p = (p_1, ..., p_n)$, $z_j(p)$ may not be the equilibrium, that is

Total Demand=Total Supply

or $z_j(p) = 0, j = 1, 2, ..., n$
If there is a price $p^* = (p_1^*, ..., p_n^*)$ and distribution $x^{i^*} = x^i(p^*, p^*w^i)$, here, x^{i^*} is the optimum solution of $(P_i), i = 1, ..., m$, such that

$$z(p^*) = \Sigma_{i=1}^m [x^i(p^*, p^*w^i) - w^i] \leq 0$$

which means that the total demand does not exceed the total supply, then we call this combination

of price and distribution (p^*, x^*) a Walras Equilibrium of this economic system. p^* is called equilibrium price and x^* equilibrium distribution.
Obviously, the Walras Equilibrium is an attainable distribution, since its total demand does not exceed its total supply.