Stat581 HW3 Solutions

7.(3') Coupon collecting. Suppose there are N different types of coupons available when buying cereal; each box contains one coupon and the collector is seeking to collect one of each in order to win a prize. After buying n boxes, what is the probability P_n that the collector has at least one of each type?

Hint: similar to example 2.1.2.

Let A_i be the event "ith type of coupon is not collected after buying n boxes", $i = 1,\dots,N$

$$P(A_i) = \frac{(N-1)^n}{N^n}, P(A_iA_j) = \frac{(N-2)^n}{N^n}, i \neq j, \dots$$

by inclusion - exclusion,

$$P(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} P(A_i) - \sum_{1 \le i < j \le N} P(A_iA_j) + \dots + (-1)^{N+1} P(\bigcap_{i=1}^{N} A_i)$$

$$= \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \frac{(N-k)^n}{N^n}$$

$$\text{so, } P_n = 1 - P(\bigcup_{i=1}^{N} A_i) = 1 - \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \frac{(N-k)^n}{N^n}$$

8. (2') We know that $P_1 = P_2$ on $\mathcal{B}$ if $P_1 = P_2$ on $\mathcal{C}$, provided that $\mathcal{C}$ generates $\mathcal{B}$ and is a π-system. Show this last property cannot be omitted.

Proof : by counter example,

Follow the example, $\Omega = \{a,b,c,d\}$ with $P_1(\{a\}) = P_1(\{d\}) = P_2(\{b\}) = P_2(\{c\}) = \frac{1}{6}$

$$P_1(\{b\}) = P_1(\{c\}) = P_2(\{a\}) = P_2(\{d\}) = \frac{1}{3}.$$

Set $\mathcal{C} = \{\{a,b\},\{d,c\},\{a,c\},\{b,d\}\}$, then $P_1 = P_2$ on $\mathcal{C}$,

clearly, $\mathcal{C}$ is not a π – system, now we want a example of some collection $\mathcal{B}$ generated by $\mathcal{C}$ but $P_1 \neq P_2$ on $\mathcal{B}$,

consider $\mathcal{B} = \sigma(\mathcal{C}) = \{\varnothing, \Omega, \{a,b\},\{d,c\},\{a,c\},\{b,d\},\{a\},\{b\},\{c\},\{d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}, \{a,d\},\{b,c\}\}$,

$P_1(\{a\}) = \frac{1}{6} \neq P_2(\{a\}) = \frac{1}{3}$, so $P_1 \neq P_2$ on $\mathcal{B}$.

10. (3') Two events A, B on the probability space $(\Omega, \mathcal{B}, P)$ are equivalent if $P(A \cap B) = P(A) \vee P(B)$.

Proof :

$$\left.\begin{array}{l} A \cap B \subset A \Rightarrow P(A \cap B) \le P(A) \\ A \cap B \subset B \Rightarrow P(A \cap B) \le P(B) \end{array}\right\} \Rightarrow P(A \cap B) \le P(A) \wedge P(B)$$

$$\Rightarrow P(A \cap B) = P(A) \vee P(B) \le P(A) \wedge P(B)$$

$$\Rightarrow P(A \cap B) = P(A) = P(B)$$

$$A \Delta B = AB^c \cup A^cB = (A \cup A^cB) \cap (B^c \cup A^cB) = (A \cup B) \cap (A^c \cup B^c) = (A \cup B) \setminus (A \cap B)$$

$$\begin{aligned} P(A \Delta B) &= P(A \cup B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) - P(A \cap B) \\ &= 0 \end{aligned}$$

11. (3') Suppose $\{B_n, n \ge 1\}$ are events with $P(B_n) = 1$ for all n. Show $P(\bigcap_{n=1}^{\infty} B_n) = 1$.

Proof :

$$P(B_n) = 1 \Rightarrow P(B_n^c) = 0 \Rightarrow \sum_{n=1}^{\infty} P(B_n^c) = 0 \Rightarrow P(\bigcup_{n=1}^{\infty} B_n^c) \le \sum_{n=1}^{\infty} P(B_n^c) = 0 \Rightarrow 1 - P(\bigcap_{n=1}^{\infty} B_n) = 0 \Rightarrow P(\bigcap_{n=1}^{\infty} B_n) = 1$$

13. (3') If $\{B_k\}$ are events such that $\sum_{k=1}^{n} P(B_k) > n-1$, then $P(\bigcap_{k=1}^{n} B_k) > 0$.

Proof :

$$\sum_{k=1}^{n} P(B_k) > n-1 \Rightarrow 1 > \sum_{k=1}^{n} [1 - P(B_k)] \Rightarrow P(\bigcup_{k=1}^{n} B_k^c) \le \sum_{k=1}^{n} P(B_k^c) < 1 \Rightarrow P(\bigcap_{k=1}^{n} B_k) = P[(\bigcup_{k=1}^{n} B_k^c)^c] > 0$$

24. (3') Suppose λ_2 is the uniform distribution on the unit square $[0,1]^2$ defined by its distribution function $\lambda_2([0,\theta_1]\times[0,\theta_2])=\theta_1\theta_2,(\theta_1,\theta_2)\in[0,1]^2$.

(a) Prove that λ_2 assigns 0 probability to the boundary of $[0,1]^2$.

Proof :

for bottom boundary B_b, $\lambda_2(B_b)=\lambda_2([0,\theta_1]\times[0,0])=\theta_1\times 0=0,\theta_1\in[0,1]$;

for left boundary B_l, $\lambda_2(B_l)=\lambda_2([0,0]\times[0,\theta_2])=0\times\theta_2=0,\theta_2\in[0,1]$;

for top boundary B_t, $\lambda_2(B_t)=\lim_{\varepsilon\to 0}\downarrow\lambda_2([0,\theta_1]\times[1-\varepsilon,1])=\lim_{\varepsilon\to 0}\downarrow\left[\lambda_2([0,\theta_1]\times[0,1])-\lambda_2([0,\theta_1]\times[0,1-\varepsilon])\right]$

$$=\theta_1-\lim_{\varepsilon\to 0}\uparrow\lambda_2([0,\theta_1]\times[0,1-\varepsilon])=\theta_1-\theta_1=0,\theta_1\in[0,1];$$

for right boundary B_r, $\lambda_2(B_r)=\lim_{\varepsilon\to 0}\downarrow\lambda_2([1-\varepsilon,1]\times[0,\theta_2])=\theta_2-\theta_2=0,\theta_2\in[0,1]$.

(b) Calculate $\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2:\theta_1\wedge\theta_2>\frac{2}{3}\}$.

$$\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2:\theta_1\wedge\theta_2>\frac{2}{3}\}=\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2:\theta_1>\frac{2}{3},\theta_2>\frac{2}{3}\}$$

$$=\lambda_2([\frac{2}{3},1]\times[\frac{2}{3},1])$$

$$=\lambda_2([0,1]\times[0,1])-\lambda_2([0,\frac{2}{3}]\times[0,1])-\lambda_2([0,1]\times[0,\frac{2}{3}])+\lambda_2([0,\frac{2}{3}]\times[0,\frac{2}{3}])$$

$$=\frac{1}{9}.$$

(c) Calculate $\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2:\theta_1\wedge\theta_2\le x,\theta_1\wedge\theta_2\le y\}$.

$$\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2:\theta_1\wedge\theta_2\le x\wedge y\}=1-\lambda_2\{(\theta_1,\theta_2)\in[0,1]^2:\theta_1\wedge\theta_2>x\wedge y\}$$

$$=1-(1-x\wedge y)^2$$

26. (3') If $A_1,\ldots,A_n$ are events, define

$$S_1=\sum_{i=1}^{n}P(A_i)$$

$$S_2=\sum_{1\le i<j\le n}P(A_iA_j)$$

$$\vdots$$

(a) Show the probability $p(m)=P[\sum_{i=1}^{n}1_{A_i}=m]$ of exactly m of the events occurring is $p(m)=S_m-\binom{m+1}{m}S_{m+1}+\ldots\pm\binom{n}{m}S_n$. Verify that the inclusion-exclusion formula is a special case of this.

Proof :

$$p(m)=\sum_{\binom{n}{m}\text{cases}}P\left[(\bigcap_{k=1}^{m}A_{i_k})\cap(\bigcup_{j=\{1,\ldots,n\}\setminus\{i_k\}}A_{i_j})^c\right]$$

$$=\sum_{\binom{n}{m}\text{cases}}P(\bigcap_{k=1}^{m}A_{i_k})-\sum_{\binom{n}{m}\text{cases}}P[(\bigcap_{k=1}^{m}A_{i_k})\cap(\bigcup_{j=\{1,\ldots,n\}\setminus\{i_k\}}A_{i_j})]$$

$$=S_m-\sum_{\binom{n}{m}\text{cases}}P\left[\bigcup_{j=\{1,\ldots,n\}\setminus\{i_k\}}[(\bigcap_{k=1}^{m}A_{i_k})\cap A_{i_j}]\right]$$

$$\overset{in-ex}{=}S_m-\sum_{\binom{n}{m}\text{cases}}\left[\sum_{j=\{1,\ldots,n\}\setminus\{i_k\}}P[(\bigcap_{k=1}^{m}A_{i_k})\cap A_{i_j}]-\sum_{j1,j2\in\{1,\ldots,n\}\setminus\{i_k\}}P[(\bigcap_{k=1}^{m}A_{i_k})\cap A_{i_{j1}}\cap A_{i_{j2}}]+\ldots\pm P[\bigcap_{i=1}^{n}A_i]\right]$$

$$=S_m-\binom{m+1}{m}S_{m+1}+\ldots\pm\binom{n}{m}S_n$$

verify inclusion – exclusion:

$$P(\bigcup_{i=1}^{n} A_i) = p(1) + \ldots + p(n)$$

$$= S_1 - \binom{2}{1} S_2 + \binom{3}{1} S_3 - \ldots \pm \binom{n}{1} S_n$$

$$+ \qquad S_2 - \binom{3}{2} S_3 + \ldots \pm \binom{n}{2} S_n$$

$$+ \ldots$$

$$+ \qquad S_n$$

$$= (-1)^{k+1} S_k \left[\sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \right]$$

$$= (-1)^{k+1} S_k \left[\sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} + 1 \right]$$

$$= (-1)^{k+1} S_k$$

(b) Referring to example 2.1.2, compute the probability of exactly m coincidences.

$$p(m) = S_m - \binom{m+1}{m} S_{m+1} + \ldots \pm \binom{n}{m} S_n$$

$$= \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \binom{n}{k} \frac{(n-k)!}{n!}$$

$$= \sum_{k=m}^{n} (-1)^{k-m} \frac{k!}{m!(k-m)!} \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!}$$

$$= \sum_{k=m}^{n} \frac{(-1)^{k-m}}{m!(k-m)!}$$

$$\approx \frac{1}{m!} e^{-1}$$