Stat581 HW7 Solutions

7.(1') Suppose $X_n, n \geq 1$ and X are uniformly bounded random variables; i.e., there exists a constant K such that

$$|X_n| \bigvee |X| \leq K.$$

If $X_n \to X$ as $n \to \infty$, show by means of dominated convergence that

$$E|X_n - X| \to 0.$$

Proof:

Since $K \in L_1$, DCT can apply, and note that $|X_n - X| \leq 2K \in L_1$.

8.(3') Suppose $X, X_n, n \geq 1$ are random variables on the space $(\Omega, \mathcal{B}, P)$ and assume

$$\sup_{\substack{\omega \in \Omega \\ n \geq 1}} |X_n(\omega)| < \infty;$$

that is, the sequence $\{X_n\}$ is uniformly bounded.

(a) Show that if in addtion

$$\sup_{\omega \in \Omega} |X(\omega) - X_n(\omega)| \to 0, n \to \infty,$$

then $E(X_n) \to E(X)$.

Proof:

Since $\{X_n\}$ is uniformly bounded and $\sup_{\omega \in \Omega} |X(\omega) - X_n(\omega)| \to 0 \Rightarrow X_n \to X$, by DCT, $E(X_n) \to E(X)$.

(b) Use Egorov's theorem to prove: If $\{X_n\}$ is uniformly bounded and $X_n \to X$, then $E(X_n) \to E(X)$.

Proof:

$$\begin{aligned}
X_n \to X & \stackrel{Egorov}{\Longrightarrow} & \exists \Lambda_\epsilon, P(\Lambda_\epsilon) < \epsilon, \sup_{\omega \in \Lambda \setminus \Lambda_\epsilon} |X(\omega) - X_n(\omega)| \to 0 (n \to \infty) \\
& \Rightarrow & \forall \epsilon > 0, \forall \omega \in \Lambda \setminus \Lambda_\epsilon, |X(\omega) - X_n(\omega)| \leq \epsilon \\
& \Rightarrow & E\Big[|X - X_n| 1_{[\omega \in \Lambda \setminus \Lambda_\epsilon]}\Big] \leq \epsilon P(\omega \in \Lambda \setminus \Lambda_\epsilon) \leq \epsilon \\
& \Rightarrow & E\Big[|X - X_n|\Big] = E\Big[|X - X_n| 1_{[\omega \in \Lambda \setminus \Lambda_\epsilon]}\Big] + E\Big[|X - X_n| 1_{[\omega \in \Lambda_\epsilon]}\Big] \leq \epsilon + 2M\epsilon \\
& \Rightarrow & E\Big[|X - X_n|\Big] \Rightarrow 0 \\
& \Rightarrow & E(X_n) \to E(X)
\end{aligned}$$

9.(2') Use Fubini's theorem to show for a distribution function $F(x)$

$$\int_{\mathbb{R}} (F(x+a) - F(x)) dx = a,$$

where dx can be interpreted as Lebesgue measure.

Proof:

$$\begin{aligned}
\int_{\mathbb{R}} (F(x+a) - F(x)) dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[\omega \in (x, x+a]]} d\omega dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[\omega \in (x, x+a]]} dx d\omega \\
&= \int_{\mathbb{R}} a d\omega \\
&= a.
\end{aligned}$$

13.)(2') Suppose the probability space is the Lebesgue interval

$$(\Omega = [0,1], \mathcal{B}([0,1]), \lambda)$$

and define

$$X_n = \frac{n}{\log n} 1_{(0,\frac{1}{n})}.$$

Show $X_n \to 0$ and $E(X_n) \to 0$ even though the condition of domination in DCT fails.

Proof:

$\forall \omega \in \Omega, \exists N$, s.t., for $n > N, 1_{(0<\omega<\frac{1}{n})} = 0$, so $X_n(\omega) = 0$, therefore $X_n \to 0$.

$E(X_n) = \frac{n}{\log n} P[(0, \frac{1}{n})] = \frac{n}{\log n}\frac{1}{n} = \frac{1}{\log n} \to 0.$

Clearly, X_n is not bounded by any r.v. in L_1.

14.(2') Suppose $X \perp\!\!\!\perp Y$ and $h : \mathbb{R}^2 \mapsto [0, \infty)$ is measurable. Define

$$g(x) = E(h(x, Y))$$

and show

$$E(g(X)) = E(h(X, Y)).$$

Proof:

$$\begin{aligned}
E(g(X)) &= \int_{\mathbb{R}} g(x) F_X(dx) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x, y) F_Y(dy) F_X(dx) \\
&= \int_{\mathbb{R}^2} h(x, y) F_Y(dy) F_X(fx) \\
&= \int_{\mathbb{R}^2} h(x, y) F_Y \times F_X(dy \times dx), \text{ by } X \perp\!\!\!\perp Y \\
&= E(h(X, Y))
\end{aligned}$$